Secondary

MATHEMATICS

Students' Book Three

Third Edition

Consider Mr Mbuyu who secured a loan of Ksh 3 780 000 from Maendeleo Building Society for the purchase of a residential house. The amount was repayable in fourteen years at a compound interest rate of 14% per annum.

This means he had to pay a monthly instalm

KENYA LITERATURE BUREAU

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CONTENTS

Chapter One

QUADRATIC EXPRESSIONS AND EQUATIONS (II)

1.1: Factorisation of Quadratic Expressions

In chapter 15 of book 2, we learnt how to factorise a quadratic expression. Study the following quadratic expressions and factorise them:

(i) $x^2 + 6x + 9$ (ii) $x^2 - 5x + 6$

(iii) $x^2 - 8x + 16$ (iv) $x^2 + x - 12$

(v) $4x^2 - 12x + 9$ (vi) $4x^2 + 4x - 3$

(vii) $9x^2 - 6x + 1$ (viii) $3x^2 - 11x - 4$

What do you notice about expressions (i), (iii), (v) and (vii) as compared to (ii), (iv), (vi) and (vii)?

Expressions in the first group, i.e., (i), (iii), (v) and (vii) factorise into two equal factors. Such expressions are called **perfect squares**. The rest are not perfect squares.

Exercise 1.1

Factorise the following expressions and hence identify the perfect squares:

1. (a) $x^2 - 6x + 8$ (b) $x^2 + 12x + 36$
2. (a) $2x^2 + x - 6$ (b) $4x^2 + 12x + 9$
3. (a) $x^2 - 8x - 9$ (b) $x^2 - 14x + 49$
4. (a) $2x^2 + 3x + 1$ (b) $16x^2 - 8x + 1$
5. (a) $9x^2 + 12x + 4$ (b) $5x^2 + 14x - 3$

1.2: Completing the Square

In section 1.1, you factorised the following expressions and got the factors given:

(i) $x^2 + 6x + 9 = (x + 3)^2$ (ii) $4x^2 - 12x + 9 = (2x - 3)^2$

(iii) $x^2 - 8x + 16 = (x - 4)^2$ (iv) $9x^2 - 6x + 1 = (3x - 1)^2$

These expressions are perfect squares. We shall now see ways of making any quadratic expression a perfect square.

Any quadratic equation can be simplified and presented in the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. We shall first deal with cases where $a = 1$, as in (i) and (iii) above. Identify similar expressions which are perfect squares from exercise 1.1. You will notice that in all cases, $\left(\frac{b}{2}\right)^2 = c$. We shall use this relation to make expressions perfect squares.

Example 1

What must be added to $x^2 + 10x$ to make it a perfect square?

Solution

Let the number to be added be a constant c. Then;

$x^2 + 10x + c$ is a perfect square.

Therefore, $(10 \div 2)^2 = c$

$\Rightarrow c = 25$

Thus, 25 must be added to $x^2 + 10x$ to make it a perfect square.

Alternatively;

Let the number to be added be a constant c.

Therefore;

$x^2 + 10x + c = (x + k)^2$, where k is another constant.

So, $x^2 + 10x + c = x^2 + 2kx + k^2$.

Comparing coefficients of x;

$2k = 10$

Therefore, $k = 5$

Comparing constant terms;

$c = k^2$

But $k = 5$.

Therefore $c = 5^2$

$= 25$

Thus, 25 must be added to $x^2 + 10x$ to make it a perfect square.

Example 2

What must be added to $x^2 +$ — $+ 36$ to make it a perfect square?

Solution

Let the term to be added be bx, where b is a constant.

Then $x^2 + bx + 36$ is a perfect square.

Therefore, $\left(\frac{b}{2}\right)^2 = 36$

Taking square roots of both sides;

$\frac{b}{2} = \sqrt{36}$

$\frac{b}{2} = \pm 6$

$b = 12$ or -12 .

Thus, the term to be added is 12x or −12x.

Alternatively;

Let the term to be added be bx, where b is a constant. Then;

$x^2 + bx + 36 = (x + m)^2$, where m is another constant.

So, $x^2 + bx + 36 = x^2 + 2mx + m^2$

Comparing the constants;

$m^2 = 36$

$m = \pm 6$

Comparing the coefficients of x;

$b = 2m$

when $m = 6, b = 12$

and when $m = -6, b = -12$

The term to be added is therefore 12x or −12x.

In each of the following expressions, insert the term which will make it a perfect square:

(i) $x^2 - 16x + —$
(ii) $x^2 + 2x + —$
(iii) $x^2 + — + 16$
(iv) $x^2 - — + 100$
(v) $A^2 - — + B^2$
(vi) $A^2 + — + B^2$

We shall now consider situations where $a \neq 1$, and is not zero, as in;

$4x^2 - 12x + 9 = (2x - 6)^2$

$9x^2 + 6x + 1 = (3x + 1)^2$

Identify similar expressions in exercise 1.1.

You will notice that in all cases, $\left(\frac{b}{2}\right)^2 = ac$.

We shall use this relation to make expressions perfect squares.

Example 3

What must be added to $25x^2 + — + 9$ to make it a perfect square?

Solution

Let the term to be added be bx.

Then, $25x^2 + bx + 9$ is a perfect square.

Therefore, $\left(\frac{b}{2}\right)^2 = 25 \times 9$

$\left(\frac{b}{2}\right)^2 = 225$

$\frac{b}{2} = \pm 15$

So, b = 30 or −30.

The term to be added is thus 30x or −30x.

Alternatively;

Let the term to be added be bx.

Therefore, $25^2 + bx + 9 = (5x + n)^2$, for some constant n.

By expanding the expression on the right hand side and comparing the constants and the coefficients of x, you should find that the term to be added is either 30x or –30x.

Example 4

What must be added to — – 40x + 25 to make it a perfect square?

Solution

Let the missing term be ax^2. Then, $ax^2 - 40x + 25$ is a perfect square.

Therefore, $(-40 \div 2)^2 = 25a$

$400 = 25a$

$a = 16$

Thus, the missing term is $16x^2$

Alternatively;

Let the missing term be ax^2. Therefore;

$ax^2 - 40x + 25 = (mx + 5)^2$, where m is another constant.

$ax^2 - 40x + 25 = m^2x^2 + 10mx + 25$

Comparing terms gives;

$10m = -40$

$m = -4$

Now $a = m^2$

$= (-4)^2$

$= 16$

Thus, the term to be added is $16x^2$.

In each of the following expressions, insert the term which will make it a perfect square.

(i) $25x^2 + 40x +$ — (ii) $9x^2 - 6x +$ —

(iii) $4x^2 +$ — $+ 25$ (iv) $36x^2 -$ — $+ 25$

(v) — $+ 14x + 1$ (vi) — $- 48x + 9$

The process of adding a term to an expression to make it a perfect square is called **completing the square**.

Exercise 1.2

Add the missing terms to the following expressions to make them perfect squares.

1. (a) $x^2 + 8x +$ — (b) $x^2 + 14x +$ — (c) $x^2 - 22x +$ —

2. (a) — $+ 2x + 1$ (b) — $- 18x + 81$ (c) $16x^2 +$ — $+ 36$

3. (a) $49x^2 - \text{—} + 4$ (b) $\frac{1}{4}x^2 + \text{—} + 1$ (c) $121x^2 + 22x + \text{—}$
4. (a) $\frac{1}{64}x^2 - \text{—} + \frac{1}{4}$ (b) $a^2x^2 + 6ax + \text{—}$ (c) $x^2y^2 + \text{—} + 4$
5. (a) $a^2x^2 + 6abx + \text{—}$ (b) $(x - 1)^2 + \text{—} + (y - 1)^2$ (c) $-10a + 1$
6. (a) $\text{—} - 60x + 25$ (b) $\text{—} - 4x + 1$ (c) $\text{—} + 2k + 4$
7. (a) $4 - 20x + \text{—}$ (b) $\text{—} + 42x + 9$

1.3: Solution of Quadratic Equations by Completing the Square

Consider the quadratic equation;

$x^2 + 4x - 12 = 0$

On adding 12 to both sides, we have;

$x^2 + 4x = 12.$

The expression on the left hand side can be made a perfect square by adding 4. Adding 4 to both sides, we get;

$x^2 + 4x + 4 = 16$

Factorising the L.H.S. gives;

$(x + 2)^2 = 16$

Taking square roots of both sides gives $x + 2 = \sqrt{16}$

$x + 2 = \pm 4$

$x = -2 \pm 4$

$x = 2$ or -6

We have solved the quadratic equation above by making use of an expression which was a perfect square. This method of solving a quadratic equation is known as **completing the square**.

Example 5

Solve $x^2 + 5x + 1 = 0$ using completing the square method.

Solution

Subtracting one from both sides and proceeding to complete the square on the left hand side;

$x^2 + 5x = -1$

$x^2 + 5x + \left(\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2 - 1$

$x^2 + 5x + \frac{25}{4} = \frac{21}{4}$

Factorising the L.H.S. gives $\left(x + \frac{5}{2}\right)^2 = \frac{21}{4}$

Taking square roots gives $x + \frac{5}{2} = \pm \sqrt{\frac{21}{4}}$

$$x = -\tfrac{5}{2} \pm \frac{4.583}{2}$$

$$= -\frac{0.417}{2} \text{ or } -\frac{9.583}{2}$$

Therefore, $x = -0.2085$ or -4.792.

Note:

The method of completing the square enables us to solve quadratic equations which cannot be solved by factorisation.

Exercise 1.3

Use the method of completing the square to solve the following quadratic equations:

1. (a) $x^2 + 2x - 1 = 0$ (b) $x^2 - 5x + 2 = 0$ (c) $x^2 + 6x + 3 = 0$
2. (a) $x^2 - 8x + 13 = 0$ (b) $x^2 - 3x - 5 = 0$ (c) $x^2 - 22x + 6 = 0$
3. (a) $x^2 + 5x + 3 = 0$ (b) $x^2 - 8x - 30 = 0$ (c) $x^2 + 7x + 3 = 0$
4. $x^2 - \frac{1}{2}x - \frac{1}{3} = 0$

If the coefficient of x^2 is not equal to 1, i.e., $a \neq 1$, proceed as in example 6 below.

Example 6

Solve the equation $2x^2 + 4x + 1 = 0$ using the method of completing the square.

Solution

Subtracting 1 from both sides;

$2x^2 + 4x = -1$

Making the coefficient of x^2 one by dividing through by 2;

$x^2 + 2x = -\frac{1}{2}$

Adding 1 to complete the square on L.H.S., $x^2 + 2x + 1 = 1 - \frac{1}{2}$

$x^2 + 2x + 1 = \frac{1}{2}$

$(x + 1)^2 = \frac{1}{2}$

$x + 1 = \pm \sqrt{\frac{1}{2}}$

$$x = -1 \pm \sqrt{0.5}$$

$$= -1 \pm 0.7071$$

Therefore, $x = 0.2929$ or -1.7071 (t 4 s.f.).

Exercise 1.3 (continued)

5. (a) $2x^2 + 3x - 7 = 0$ (b) $3x^2 + 7x - 4 = 0$ (c) $4x^2 + 12x - 9 = 0$
6. (a) $5x^2 + 6x - 3 = 0$ (b) $4x^2 + 3x - 5 = 0$ (c) $4 - 9x - 3x^2 = 0$
7. (a) $6x^2 - 5x - 4 = 0$ (b) $2x^2 + 9x + 9 = 0$ (c) $2x(x + 1) = 4$
8. $(x + 1)(x + 3) = 13$

1.4: The Quadratic Formula

Consider the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$

Let us solve this equation using the method of completing the square.

Subtracting c from both sides; $ax^2 + bx = -c$

Dividing through by a; $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Complete the square on L.H.S. by adding $\left(\frac{b}{2a}\right)^2$ to both sides;

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

but $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a}$

$$= \frac{b^2 - 4ac}{4a^2}$$

Therefore, $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Factorising the L.H.S. gives $(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$

Taking square roots of both sides gives $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

If therefore follows that $x = \frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the solution to the general quadratic equation and is known as the **quadratic formula.**

Example 7

Use the quadratic formula to solve $2x^2 - 5x - 3 = 0$.

Solution

Comparing this equation to the general equation $ax^2 + bx + c = 0$, we get;

$a = 2$

$b = -5$

$c = -3$

Substituting in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

$$x = \frac{-5 \pm \sqrt{25 - 4(2)(-3)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{49}}{4}$$

$$= \frac{5 \pm 7}{4}$$

$$= \frac{12}{4} \text{ or } \frac{-2}{4}$$

Therefore, $x = 3$ or $-\frac{1}{2}$

Exercise 1.4

Use the quadratic formula to solve the following equations:

1. (a) $x^2 + 7x + 3 = 0$ (b) $x^2 - 4x + 3 = 0$ (c) $2x^2 + 11x + 7 = 0$
2. (a) $2x + 7 - 7x^2 = 0$ (b) $3x^2 - 3x - 2 = 0$ (c) $5r^2 - 5r + 1 = 0$
3. (a) $4d^2 + 7d + 3 = 0$ (b) $9p^2 + 24p + 16 = 0$ (c) $6k^2 + 9k + 1 = 0$
4. $1 - 3x - 3x^2 = 0$

From the quadratic formula, we notice that the nature of the roots will depend on the value of $b^2 - 4ac$. There are three possibilities, i.e.:

(i) $b^2 - 4ac > 0$: Here, the value of $\sqrt{b^2 - 4ac}$ will be real and will take either a positive or negative value, leading to two distinct roots (values of x).

(ii) $b^2 - 4ac = 0$: In this case, the quadratic formula will give us a single value. The quadratic equation has repeated roots and is a perfect square.

(iii) $b^2 - 4ac < 0$: Here, $b^2 - 4ac$ will have no real values. The quadratic equation has no real roots.

The expression $b^2 - 4ac$ is known as the **discriminant**. It allows us to determine the nature of the roots of a quadratic equation.

Determine the nature of the roots of the following quadratic equations:

(i) $x^2 - 3x + 4 = 0$ (ii) $x^2 + 5x + 1 = 0$ (iii) $x^2 + 6x + 9 = 0$
(iv) $2x^2 - 3x + 1 = 0$ (v) $3x^2 + x + 4 = 0$ (vi) $(x - 4)(3x - 1) = 0$

1.5: Formation of Quadratic Equations

Consider the following examples.

Example 8

Peter travels to his uncle's home, 30 km away from his place. He cycles for two thirds of the journey before the bicycle develops mechanical problems and he has to push it for the rest of the journey. If his cycling speed is 10 km/h faster than his walking speed and he completes the journey in 3 hours 20 minutes, determine his cycling speed.

Solution

Let Peters' cycling speed be x km/h. Then, his walking speed is (x – 10) km/h.

$$\text{Time taken in cycling} = (\tfrac{2}{3} \text{ of } 30) \div x$$

$$= \tfrac{20}{x} \text{ h}$$

$$\text{Time taken in walking} = (30 - 20) \div (x - 10)$$

$$= \frac{10}{x - 10} \text{ h}$$

$$\text{Total time} = \left(\frac{20}{x} + \frac{10}{x + 10}\right)\text{h}$$

$$\text{Therefore, } \frac{20}{x} + \frac{10}{x - 10} = 3\tfrac{1}{3}$$

$$\frac{20}{x} + \frac{10}{x - 10} = \tfrac{10}{3}$$

$$60(x - 10) + 30(x) = 10(x)(x - 10)$$

$$10x^2 - 190x + 600 = 0$$

$$x^2 - 19x + 60 = 0$$

$$x = \frac{19 \pm \sqrt{361 - 240}}{2}$$

$$x = 15 \text{ or } 4$$

If his cycling speed is 4 km/h, then his walking speed is (4 – 10) km/h, which gives –6 km/h. Thus, 4 is not a realistic answer to this situation.

Therefore, his cycling speed is 15 km/h.

Example 9

A positive two-digit number is such that the product of the digits is 24. When the digits are reversed, the number formed is greater than the original number by 18. Find the number.

Solution

Let the ones digit of the number be y and the tens digit be x.

Then, $xy = 24$ ……… (1)

When the number is reversed, the ones digit is x and the tens digit is y.

Therefore;

$(10y + x) - (10x + y) = 18$

$9y - 9x = 18$

$y - x = 2$

$y = x + 2$ ………… (2)

Substituting (2) in equation (1) gives;

$x (x + 2) = 24$

$x^2 + 2x - 24 = 0$

$$x = \frac{-2 \pm \sqrt{4 + 96}}{2}$$

$= 4$ or -6

Since the required number is positive, $x = 4$ and $y = 4 + 2 = 6$.

Therefore, the number is 46.

Exercise 1.5

1. Solve the following quadratic equations using any suitable method.
    - (a) $x^2 - x - 6 = 0$
    - (b) $6x^2 + 11x + 3 = 0$
    - (c) $6x^2 + 13x + 6 = 0$
    - (d) $2x^2 + x = 0$
    - (e) $\frac{1}{2}x^2 - 2x - 2 = \frac{3}{4}(x - 2)$
    - (f) $4 - 25x^2 = 0$
    - (g) $4y^2 + \frac{1}{2}y = 0$
    - (h) $k^2 + 1.5 = 3k$
    - (i) $3y^2 - 2y - 5 = 0$
    - (j) $t^2 - 15t + 8 = 0$
    - (k) $2 - b - 10b^2 = 0$
    - (l) $\frac{x - 1}{2} + \frac{x + 3}{4} = \frac{1}{x - 1}$
    - (m) $-1 - 5r + 7r^2 = 0$
    - (n) $\frac{2}{y} + 1 = \frac{y - 2}{3}$
    - (p) $\frac{t}{6} + \frac{1}{2 - t} = \frac{1}{3}$
    - (q) $\frac{y - 3}{4} - \frac{1}{y} = \frac{4}{3} - y$
    - (r) $\frac{1}{x - 1} + 3 = \frac{1}{x + 2} - \frac{1}{4}$

2. ABCD is a rectangle whose area is 170 cm². The internal rectangle is drawn with dimensions as shown in figure 1.1.

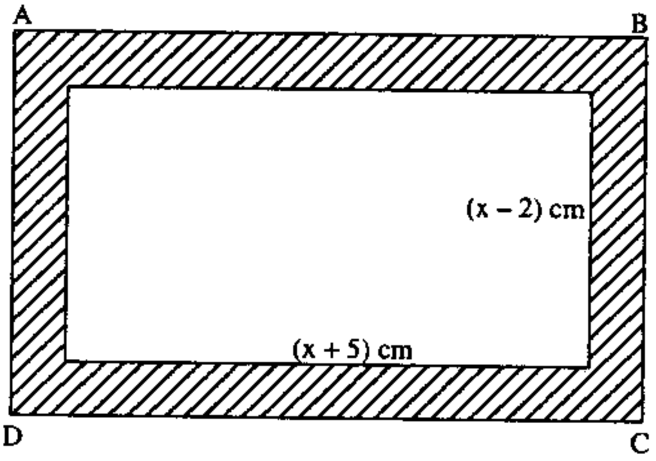


Fig. 1.1

Determine the area of the shaded part if its thickness is $\frac{1}{10}x$ cm.

3. A group of young men decided to raise sh. 480 000 to start a business. Before the actual payment was made, four of the members pulled out and each of those remaining had to pay an additional sh. 20 000. Determine the original number of members.

4. A piece of wire 90 cm long is bent into the pattern ABCDAEFD, in that order, which consists of two equal rectangles, see figure 1.2.

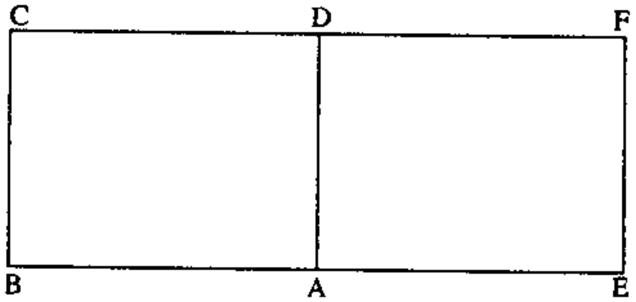


Fig. 1.2

If the total area enclosed is 300 cm², determine possible lengths of:
(a) BC.
(b) CF.

5. A lawn is in the shape of a right-angled triangle. The lengths of the two shorter sides are (x + 2) m and (4x + 4) m, while the length of the hypotenuse is 5x m. Form and solve an equation in x and hence find the area of the lawn in m².

6. A boat's speed in still water is 4 km/h. The boat cruises from A to B along

a river flowing at an average speed of x kmh in the direction A to B. If the distance AB is 5 km and the boat takes 2 hours more on its return journey, determine x. Hence, find the total time taken for the whole journey.

7. The length of a rectangle is three times its breadth. If the breadth is decreased by 2 m and the length increased by 4 m, the area of the rectangle is decreased by a third. Find the breadth of the original rectangle. Hence, find its area.
8. A two-digit number is such that the product of its digits is 12. When the digits are reversed, the number formed exceeds the original by 9. Find the original number.

1.6: Graphs of Quadratic Functions

We have drawn graphs of linear equations of the form $y = mx + c$ and noticed that each value of x gives a specific value of y. For example, in the equation $y = 2x + 1$, if $x = 2$, then $y = 5$.

Generally. if y depends on x, y is said to be a **function** of x provided that for each value of x, there is a unique value of y.

In this section, we shall be take a look at graphs of quadratic functions and their applications to the solution of equations.

Example 10

Draw the graph of $y = x^2$ for values of x from – 4 to 4.

Solution

Table 1.1 below shows the corresponding values of x and y.

Table 1.1

x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Using the values of x and y in the table, we choose suitable scales. In this case we choose: 10 small squares to represent 1 unit along the x-axis and 10 small squares to represent 5 units along the y-axis.

We plot the points and join them by a smooth curve.

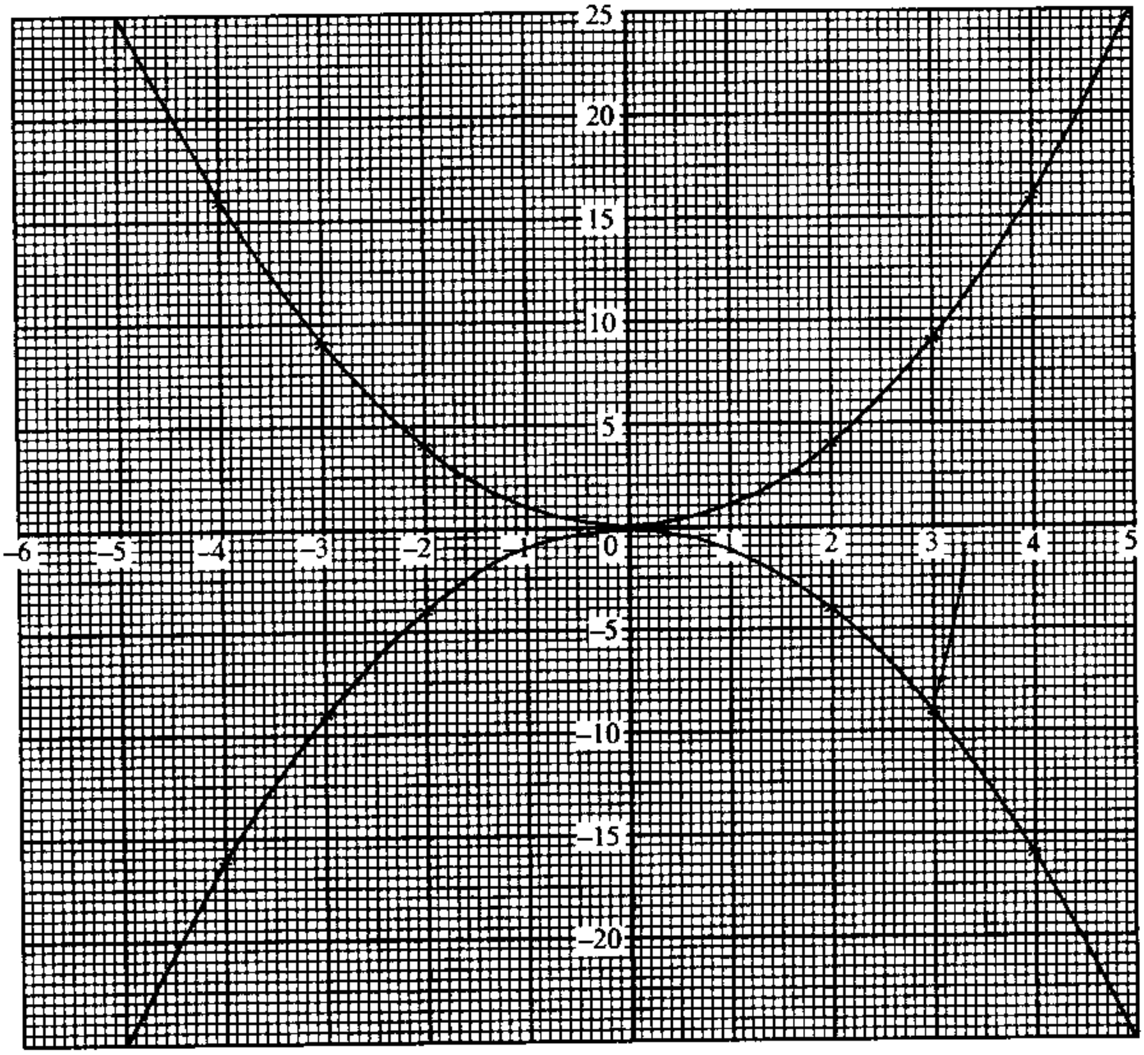


Fig. 1.3

The graph of $y = x^2$ is shown in figure 1.3.

On the same axes and using the same range, draw the graphs of $y = x^2$ and $y = -x^2$.

What is the relationship between the two curves? State the lines of symmetry of the two curves.

Note:

For the graph of $y = x^2$, the minimum value of the function is zero and for the graph of $y = -x^2$, the maximum value of the function is zero.

Exercise 1.6

1. Copy and complete tables 1.2(a) and (b) below for the functions $y = x^2 - 2x - 8$ and $y = 8 + 2x - x^2$ respectively. Hence draw the graphs of the given quadratic functions on the same axes.

Table 1.2 (a): $y = x^2 - 2x - 8$

x	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x^2	25								9				
$-2x$	10								–6				
-8	–8								–8				
$y = x^2 - 2x - 8$	27								–5				

Table 1.2 (b): $y = 8 + 2x - x^2$

x	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
8		8				8							
$+2x$		–8				0							
x^2		–16				0							
$y = 8 + 2x - x^2$		–16				8							

State the equation of the line of symmetry between the two curves.

2. Draw the graphs of the following quadratic functions for the given range of values of x. In each case, state the equation of the line of symmetry and the minimum/maximum value of the function.
   (a) $y = x^2 + 2x - 3$ $\quad -6 \leq x \leq 4$
   (b) $y = x^2 + 6x + 9$ $\quad -7 \leq x \leq 1$
   (c) $y = x^2 + 5x + 7$ $\quad -7 \leq x \leq 2$
   (d) $y = 12 - 4x - x^2$ $\quad -8 \leq x \leq 4$
   (e) $y = - x^2 + 2x - 5$ $\quad -3 \leq x \leq 5$
3. Using the same axes, draw the graphs of:
   (a) $y = x^2$ (b) $y = x^2 + 10$ (c) $y = x^2 - 10$ for $-2 \leq x \leq 4$.
   (i) Identify the minimum points.
   (ii) What maps $y = x^2$ onto each of the other functions?
   (iii) What maps $y = x^2 + 10$ onto $y = x^2 - 10$?
4. By choosing a suitable range of values of x, draw the graph of $y = (3 - x)(x - 4) + 1$. Using the same axes, draw the graph of its reflection in the x-axis. What is the equation of the line of symmetry for the two curves?
5. Copy and complete table 1.3 below and hence draw the graph of $y^2 = 4x$.

Table 1.3

y	–4	–3	–2	–1	0	1	2	3	4
x		$\frac{9}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{9}{4}$	4

1.7: Graphical Solutions of Quadratic Equations

Consider the quadratic equation $x^2 - x - 6 = 0$, whose solution is $x = 3$ or $x = -2$.

Using table 1.4 below, the graph of the quadratic function $y = x^2 - x - 6$ is drawn as shown in figure 1.4.

Table 1.4

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	24	14	6	0	–4	–6	–6	–4	0	6	14

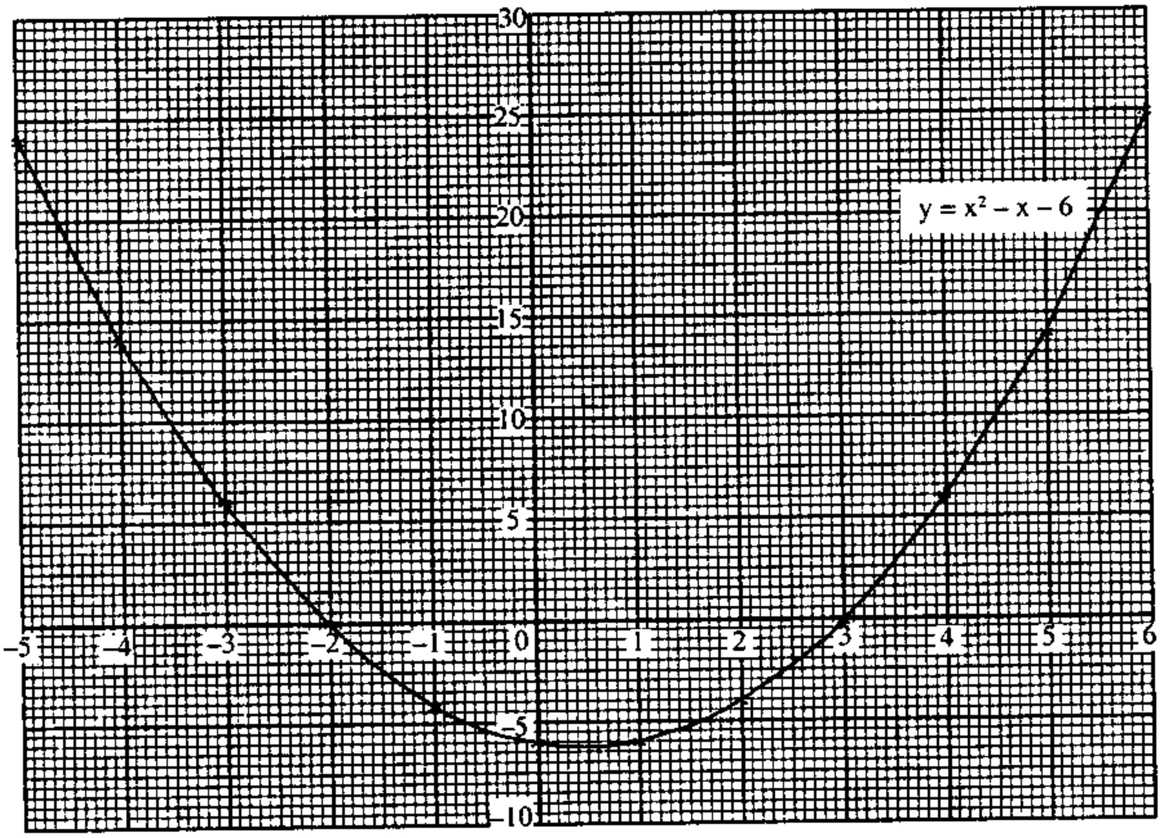


Fig. 1.4

We notice that the graph cuts the x-axis (the line $y = 0$) at $x = 3$ and $x = -2$, which are the roots of the equation $x^2 - x - 6 = 0$

In general, if the graph of $y = a,\ x^2 + bx + c$ cuts the x-axis, the values of x at the points of intersection of the curve and the x-axis are the roots of the equation $ax^2 + bx + c = 0$.

Figure 1.5 shows sketches of graphs of quadratic functions. Where possible, state.

(i) the roots of the quadratic equation.

(ii) the corresponding quadratic equation.

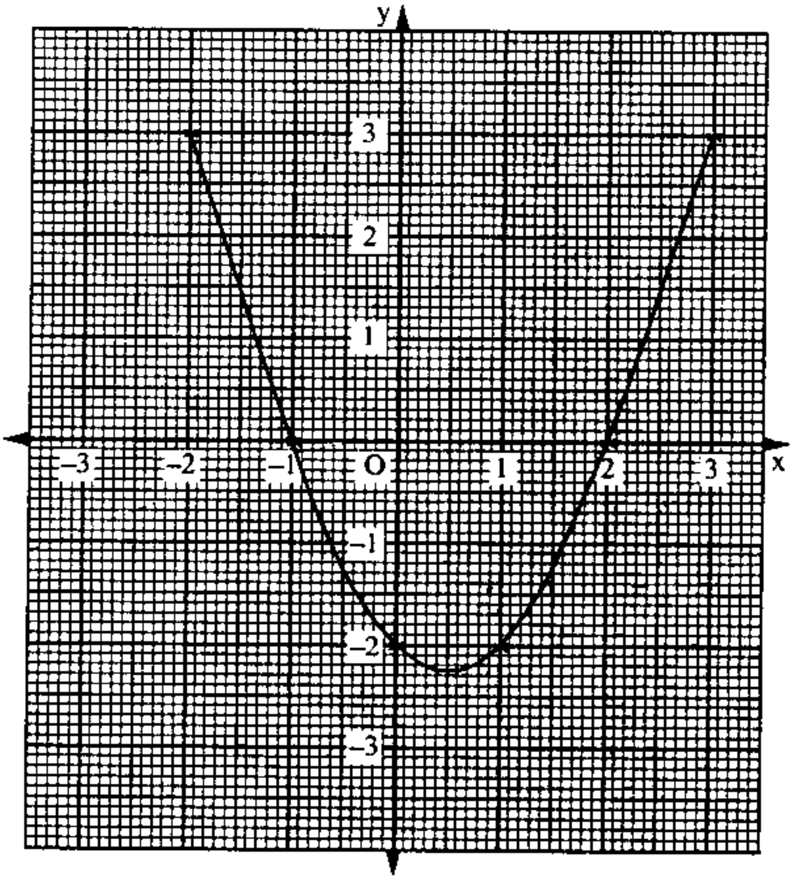


Fig. 1.5 (i)

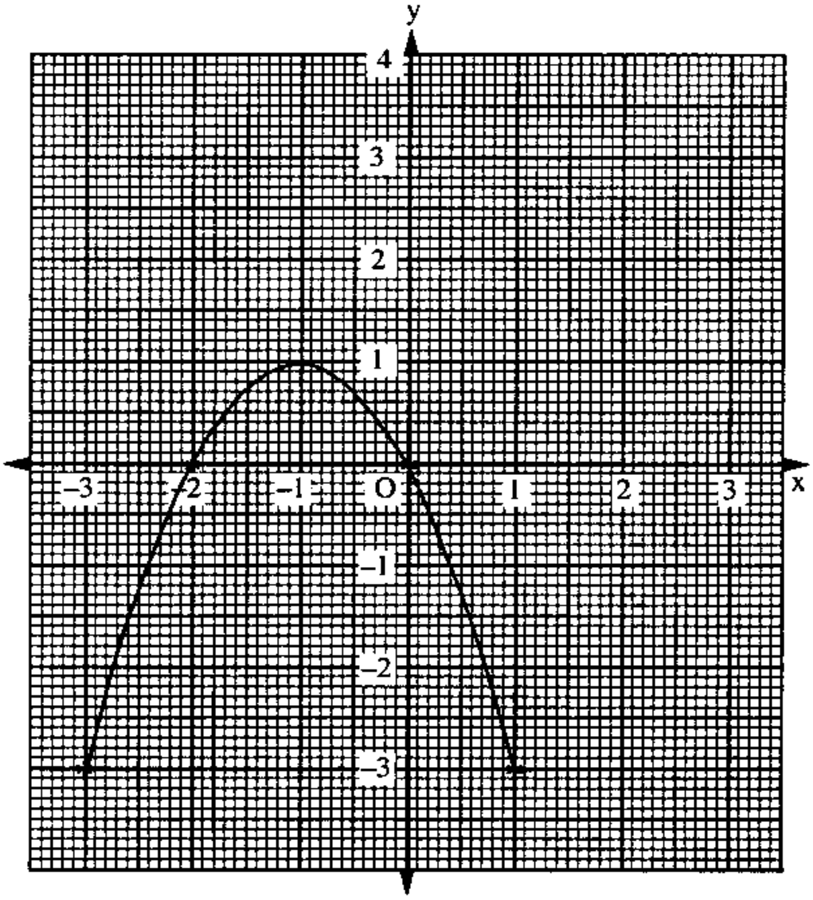


Fig. 1.5 (ii)

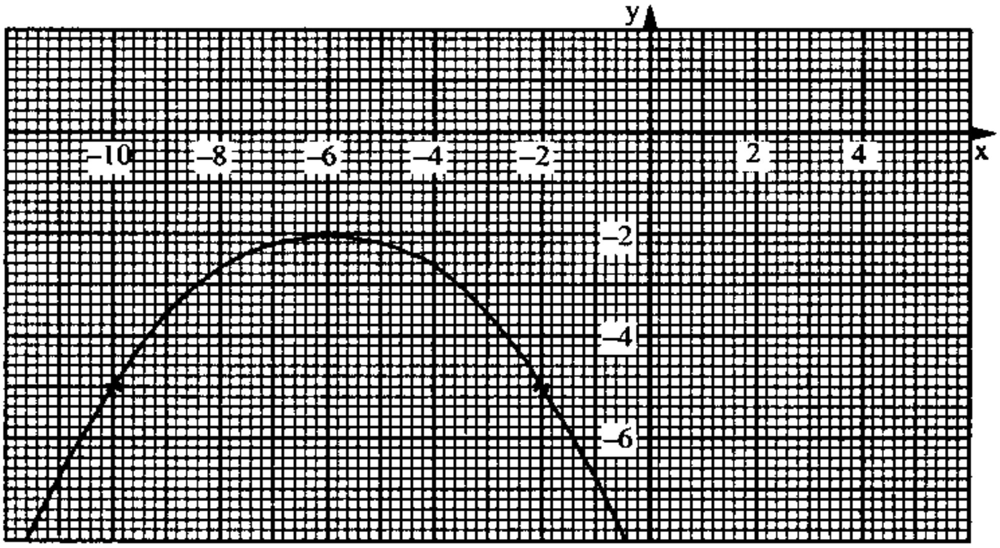


Fig. 1.5 (iii)

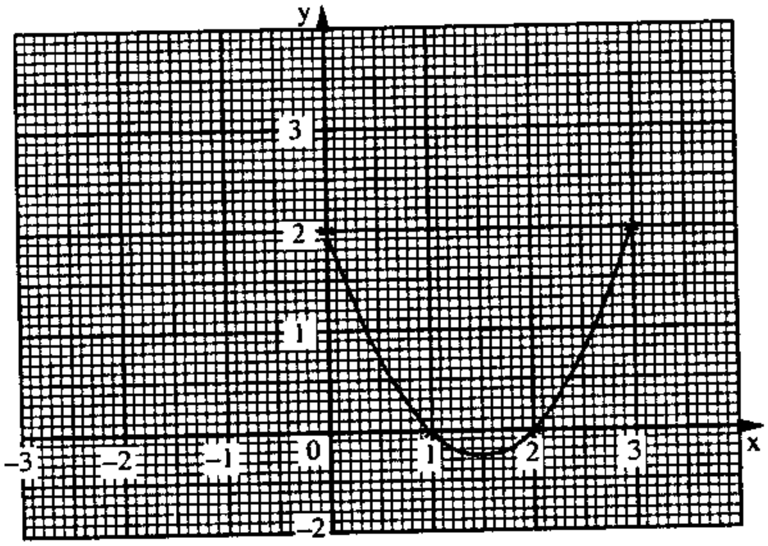


Fig. 1.5 (iv)

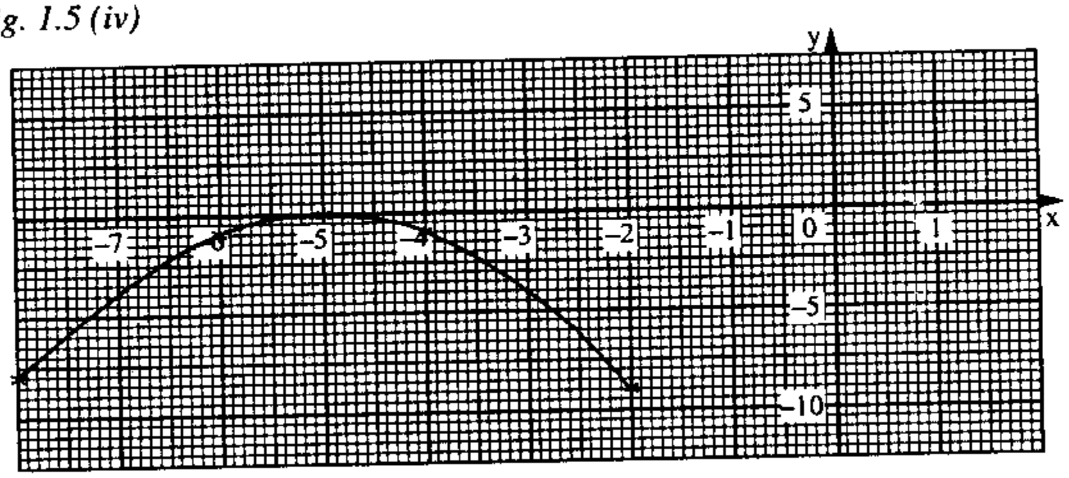


Fig. 1.5 (v)

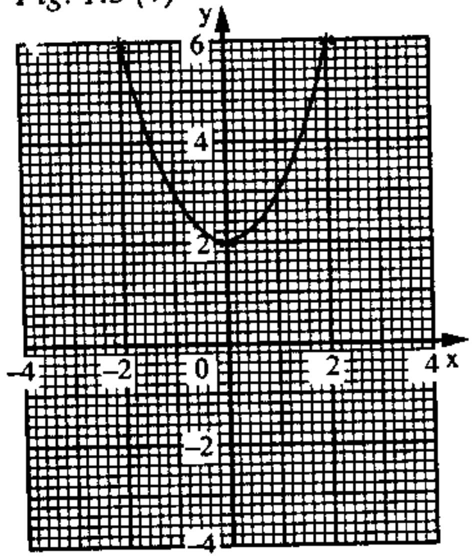


Fig. 1.5 (vi)

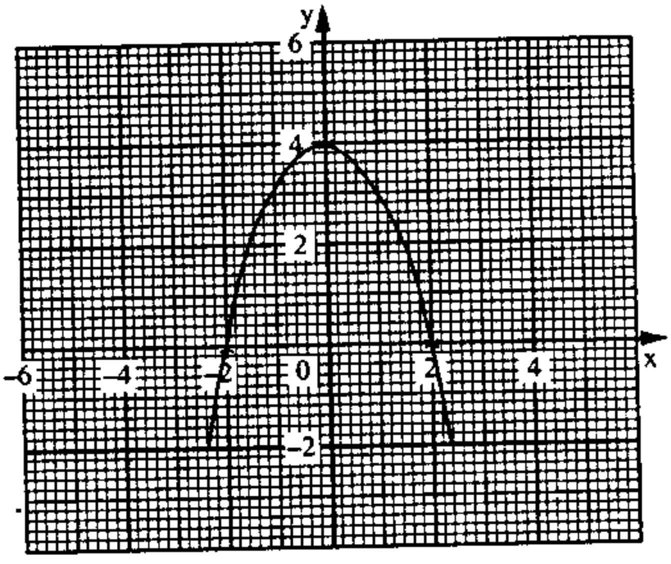


Fig. 1.5 (vii)

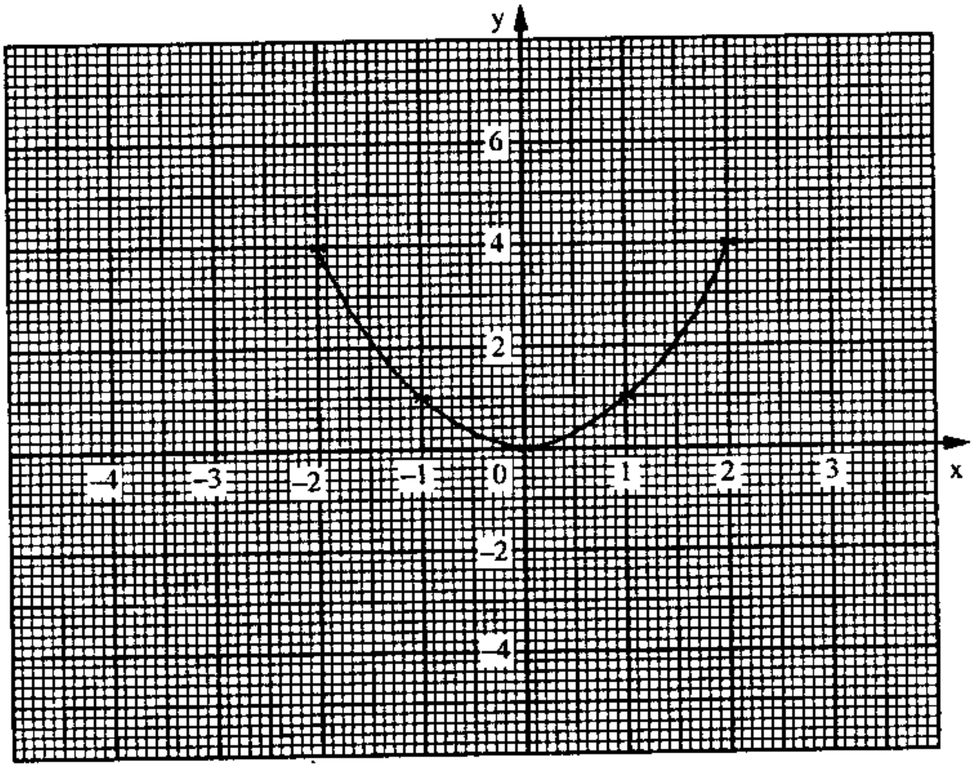


Fig. 1.5 (viii)

We notice that:

(i) If the graph cuts the x-axis at two distinct points, then the equation has two distinct roots, e.g., figure 1.5 (i).

(ii) If the graph touches the x-axis at only one point, then the equation has a repeated root at that point, e.g., figure 1.5 (viii).

(iii) The graphs in figure 1.5(iii) and (vi) neither touch nor cross the x-axis. We say that their equations have no real roots.

Exercise 1.7

1. Draw the graphs of the following quadratic functions. Where possible, estimate the roots from your graph.

(a) $y = 2x^2 - 4x + 1$ (b) $y = x^2 - x + 1$ (c) $y = x^2 + x + 1$

(d) $y = x^2 + 4x + 4$ (e) $y = 5 - 3x - x^2$ (f) $y = -3 + x - x^2$

1.8: Graphical Solutions of Simultaneous Equations

Graphical methods may be used to solve simultaneous equations. In this section, we shall consider simultaneous equations, one of which is linear and the other quadratic.

Example 11

Solve the following simultaneous equations graphically:
$y = x^2 - 2x + 1$
$y = 5 - 2x$

Solution
Table 1.5 gives corresponding values of x and y for the equation $y = x^2 - 2x + 1$

Table 1.5

x	–2	–1	0	1	2	3	4
y	9	4	1	0	1	4	9

We use the table to draw the graph of $y = x^2 -2x + 1$, as shown in figure 1.6. On the same axes, the line $y = 5 - 2x$ is drawn.

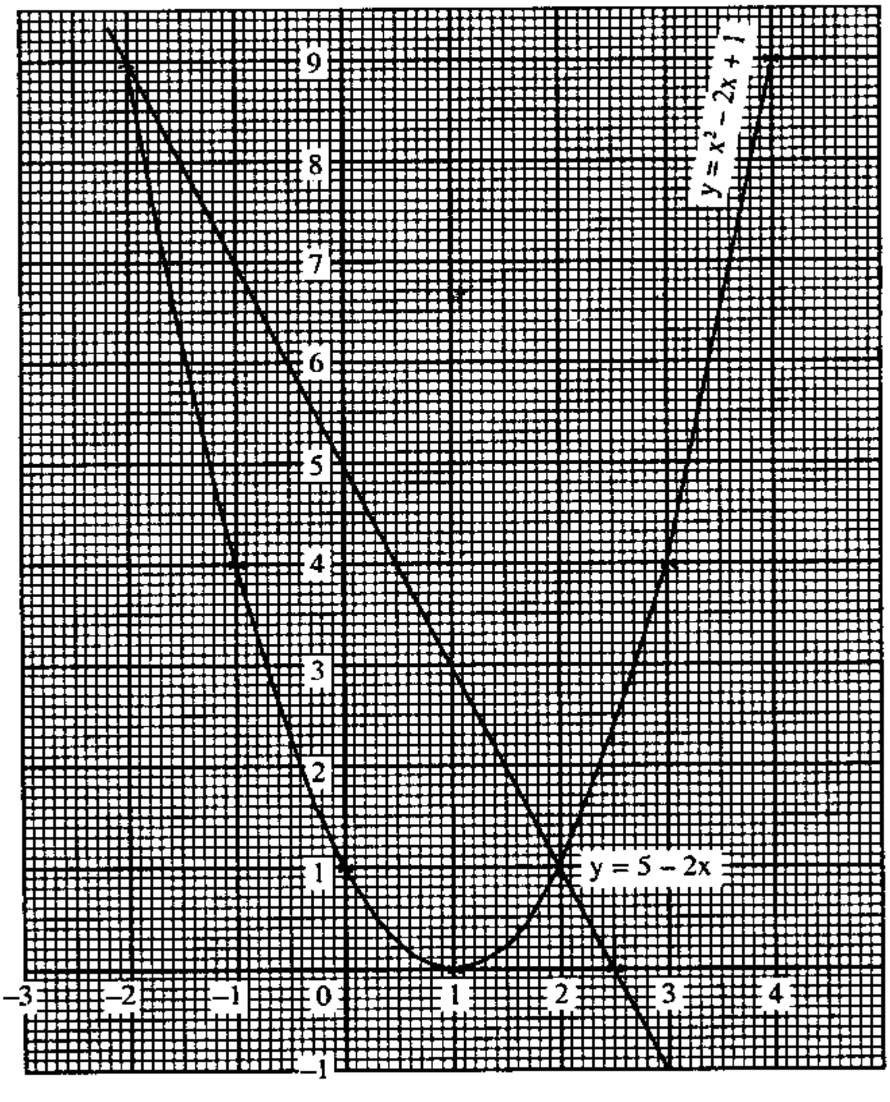


Fig. 1.6

The points where the line $y = 5 - 2x$ and the curve $y = x^2 - 2x + 1$ intersect give the solution. The points are $(-2, 9)$ and $(2, 1)$. Therefore, when $x = -2$, $y = 9$ and when $x = 2$, $y = 1$.

Exercise 1.8

1. Use the graphical method to solve the following pairs of simultaneous equations:
   (a) $y = x^2 + 3x - 6$
   $y = x + 1$
   (b) $y = x^2 + 4x + 1$
   $y = 2x + 1$
   (c) $y = 2x^2 + 3x - 5$
   $y = 2 - x$
   (d) $y = 3x^2 - 4x - 5$
   $y = 2 - 3x$
   (e) $y = \frac{x^2}{3} - \frac{2}{3}x + 2$
   $y = \frac{4}{3}x + 1$
2. Re-arrange each of the following quadratic equations to form a pair of simultaneous equations and hence draw a graph to solve the equation. (Hint: Rewrite $x^2 - 3x + 1 = 0$ as $x^2 = 3x - 1$, then draw the graphs of $y = x^2$ and $y = 3x - 1$).
   (a) $x^2 - 3x + 2 = 0$
   (b) $3x^2 - 7x + 4 = 0$
   (c) $\frac{x-2}{4} = \frac{x+2}{x}$
   (d) $\frac{x^2}{4} - \frac{7}{8}x = 1$
   (e) $x^2 - 22x + 6 = 0$
   (f) $x^2 + 7x + 3 = 0$

Further Graphical Solutions

A graph of a given quadratic function can be used to solve other related quadratic equations.

Example 12

Use the graph of the function $y = x^2 + 3x + 1$ to solve:

(a) $x^2 + 3x + 1 = 10$ (b) $x^2 + 3x - 4 = 0$ (c) $x^2 + 2x - 2 = 0$

Solution

Table 1.6 below gives corresponding values of x and y for the function $y = x^2 + 3x + 1$.

Table 1.6

x	−5	−4	−3	−2	−1	0	1	2
y	11	5	1	−1	−1	1	5	11

Figure 1.7 shows the graph of $y = x^2 + 3x + 1$.

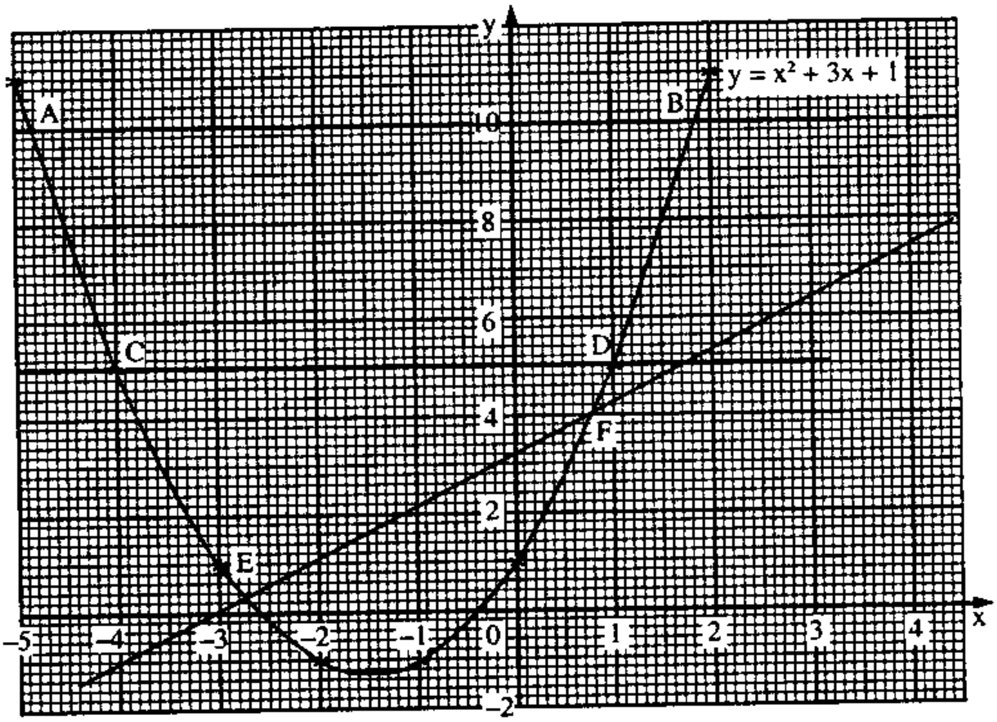


Fig. 1.7

(a) $x^2 + 3x + 1 = 10$ is the equation we want to solve. But $y = x^2 + 3x + 1$ has been drawn. Comparing the two, $y = 10$.

We draw the line $y = 10$ and read off the values of x at the points A and B on the graph where the line intersects the curve. At these points; $x = -4.9$ and $x = 1.9$.

These are the solutions of $x^2 + 3x + 1 = 10$

(b) $x^2 + 3x - 4 = 0$

Comparing this equation with the curve already drawn, i.e.,

$y = x^2 + 3x + 1$

$\underline{o = x^2 + 3x - 4}$

$y = 5$

By subtracting we can eliminate the part having x^2 and remain with a linear equation. This is the line that we will include in our curve in order to solve the quadratic equation.

Draw the line $y = 5$ and read off the values of x at the points C and D on the graph where the line intersects the curve, i.e., $x = -4$ or $x = 1$

(c) $x^2 + 2x - 2 = 0$.

Comparing this equation with the curve already drawn, we get;

$y = x^2 + 3x + 1$

$\underline{o = x^2 + 2x - 2}$

$y = x + 3$

We need to include the line $y = x + 3$ on the curve already drawn so that we can solve the quadratic equation $x^2 + 2x - 2 = 0$

We draw the line $y = x + 3$ and read off the values of x at points E and F on the graph. Thus, $x = -2.7$ or 0.8.

Exercise 1.9

For each of the following, draw the graph of the given function over the given range and use it to solve the given equations.

1. $y = x^2 + 5x + 4$ for $-7 \leq x \leq 3$
   (a) $x^2 + 5x + 4 = 0$
   (b) $x^2 + 5x + 2 = 0$
   (c) $x^2 + 4x + 3 = 0$
2. $y = 2x^2 + x - 1$ for $-4 \leq x \leq 4$
   (a) $2x^2 + x - 1 = 0$
   (b) $2x^2 + x = 0$
   (c) $2x^2 - 5 = 0$
3. $y = 4 + 6x - x^2$ for $-2 \leq x \leq 8$
   (a) $4 + 6x - x^2 = 0$
   (b) $x^2 - 4x = 0$
   (c) $x^2 - \frac{14}{3}x + 1 = 0$
4. $y = \frac{x^2}{2} - \frac{2}{3}x - 1$ for $-4 \leq x \leq 4$
   (a) $\frac{x^2}{2} - \frac{2}{3}x - 1 = 0$
   (b) $\frac{x^2}{2} - \frac{2}{3}x - 3 = 0$
   (c) $\frac{x^2}{2} - \frac{x}{2} - 2 = 0$
5. $y = 2x^2 + 5x - 12$ for $-8 \leq x \leq 4$
   (a) $2x^2 + 5x - 12 = 0$
   (b) $x^2 + x - 6 = 0$
   (c) $3 - 7x - 3x^2 = 0$

Chapter Two

APPROXIMATIONS AND ERRORS

2.1: Computing using Calculators

A calculator is an electronic device which can be used to carry out mathematical computations. These are two types of calculators, namely;

(i) simple calculators, e.g., those used by shopkeepers and cashiers.

(ii) scientific calculators which can be used for virtually all operations. In this course, the scientific calculator will be used.

On a calculator, the keypad consists of different operations and digits 0 to 9. To accommodate all operations, the same button may have two functions (one written on top, the other beside). The key which is written beside can be accessed by pressing the inverse (shift or second function) key followed by the button containing the key. The mode of operation varies from one calculator to another, thus the use of the manual is important.

Figure 2.1 shows part of the keypad of a calculator.

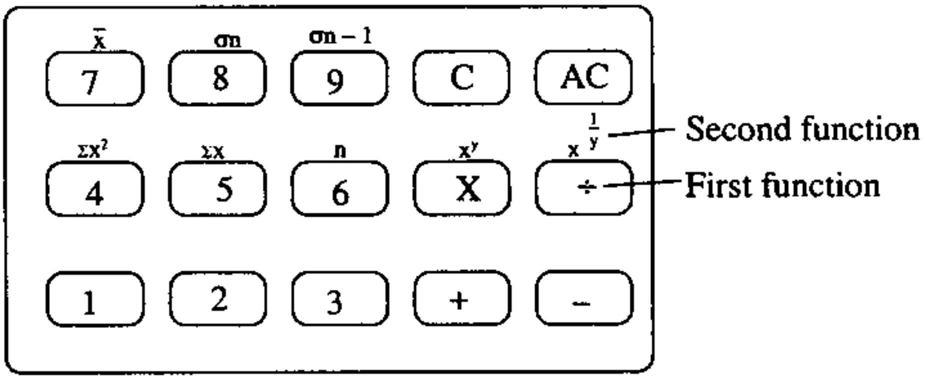


Fig. 2.1

A calculator can display as many digits as will fit on its screen. For example, to display a single digit number, e.g., 5, key in 5. To display a two-digit number like 12, key in 1 followed by 2. A negative number such as – 9 can be displayed as follows, key in 9 followed by $\boxed{+/-}$.

Use a calculator to display the following:

(i) – 8
(ii) 14
(iii) – 41
(vi) 426
(v) – 624
(vi) 5 272

The Four Basic Operations using Calculators

Example 1

5 + 8 = ?

Solution

Step I Key in 5.
Step II Key in +
Step III Key in 8
Step IV Key in =
The screen displays 13.
Therefore, 5 + 8 = 13

Example 2

–6 + 15 = ?

Solution

Step I Key in – 6
Step II Key in +
Step III Key in 15
Step IV Key in =
The screen displays 9
–6 + 15 = 9

Example 3

28 – 13 = ?
Step I Key in 28
Step II Key in –
Step III Key in 13
Step IV Key in =
The screen displays 15. Thus, 28 – 13 = 15

Example 4

4 x 9 = ?

Solution

Step I Key in 4
Step II Key in x
Step III Key in 9
Step IV Key in =
The screen displays 36. So, 4 x 9 = 36

Example 5

$35 \div 7$?

Solution

Step I Key in 35

Step II Key in $\div$

Step III Key in 7

Step IV Key in =

The screen displays 5. Therefore, $35 \div 7 = 5$

Example 6

$$\frac{3.51 + 4.51}{7.5 \times 1.28}$$

Solution

Step I Key in 3.51

Step II Key in +

Step III Key in 4.51

Step IV Key in =

Step V Key in $\div$

Step VI Key in (

Step VII Key in 7.5

Step VIII Key in x

Step IX Key in 1.28

Step X Key in)

The screen displays 0.8354166.

Be careful to press the equals key when you have completed the top line (numerator), so that the calculator displays 8.02. On most calculators, it is necessary to enclose the bottom line (denominator) in brackets, i.e., (7.5×1.28).

Squares and Squares Roots of Numbers using a Calculator

Most calculators have the keys x^2 and $\sqrt{\ }$, which can be used to find the square and square root respectively.

Example 7

Find the square of 17.

Solution

Step I Key in 17

Step II Key in x^2

The screen displays 289, which is the square of 17.

The square of 17 can also be found as below:

Step I Key in 17

Step II Key in x^y or y^x

Step III Key in 2

Example 8

Find the square root of 324.

Solution

Step I Key in 324

Step II Key in $\sqrt{\ }$

The screen displays 18, which is the square root of 324.

This can be achieved as below:

Step I Key in 324

Step II Key in $\sqrt[y]{x}$ or $x^{\frac{1}{y}}$

Step III Key in 2

Cubes and Cube Roots of Numbers using a Calculator

Cubes and cube roots can be obtained in the same way as squares and square roots.

Example 9

Use a calculator to find 5^3.

Solution

Step I Key in 5

Step II Key in x^3

The calculator displays 125, which is the cube of 5.

Alternatively;

Step I Key in 5.

Step II Key in x^y or y^x

Step III Key in 3

125 is displayed.

Example 10

Find the cube root of 64.

Solution

Step I Key in 64

Step II Key in $\sqrt[3]{\ }$

The screen displays 4, which is the cube root of 64.

Alternatively;

Step I Key in 64

Step II Key in $\sqrt[y]{x}$ or $x^{\frac{1}{y}}$

Step III Key in 3

The screen displays 4.

Exercise 2.1

1. Use a calculator to work out each of the following:

(a) (i) $26 + 53$ (ii) $267.29 + 729.78$

(iii) $-879 + 641$ (iv) $4\ 693 + 74\ 938$

(v) $-8\ 926 + 5\ 679 + (-\ 993\ 96)$ (vi) $0.003792 + 8.3698$

(b) (i) $72\ 938 - 127696$ (ii) $-0.0082 - 0.0017$

(iii) $-0.00293 - 2.695$ (iv) $824 - 749$

(v) $2\ 463 - 9\ 361 + 2\ 693$ (vi) $577 - 805 - 415$

(c) (i) 13×8.5 (ii) -25×78

(iii) 0.0045×267 (iv) -34×-18

(v) 0.0078×0.00856 (v) 529×894

(d) (i) $-536 \div 789$ (ii) $-5.0067 \div 19$

(iii) $-0.0082 \div (-\ 0.0017)$ (iv) $52 \div 16$

(v) $6\ 213 \div 923$ (vi) $359 \div 0.00394$

(e) (i) 89^2 (ii) $(-6.003)^2$ (iii) $(7.09)^2$

(iv) 349^2 (v) 46^2 (vi) $(-\ 429)^2$

(f) (i) $\sqrt{869}$ (ii) $\sqrt{89\ 361}$ (iii) $\sqrt{726}$

(iv) $\sqrt{0.0092}$ (v) $\sqrt{0.00317}$ (vi) $\sqrt{82}$

(g) (i) 68^3 (ii) $(-\ 341)^3$ (iii) $(0.06867)^3$

(iv) $(1\ 671)^3$ (v) $(0.0003947)^3$ (vi) 13^3

(h) (i) $(52)^{\frac{1}{3}}$ (ii) $\sqrt[3]{692}$ (iii) $(-673)^{\frac{1}{3}}$

(iv) $\sqrt[3]{0.0082}$ (v) $(0.00731)^{\frac{1}{3}}$ (vi) $\sqrt[3]{-0.00321}$

(i) (i) $\sqrt{\frac{693 \times 31}{44\,673}}$ (ii) $\left(\frac{536}{437 \times 620}\right)^{\frac{1}{3}}$

(iii) $\frac{1}{\sqrt{82.49}} + \frac{3}{(0.089)^2}$ (iv) $\sqrt{\frac{34^3 - 257}{97 \times 1\,243}}$

2.2: Approximation

There are a number of cases where exact figures may not be required. For instance, in a certain centre, the number of aids orphans may be given as 250. The actual number could fluctuate between 240 and 260. In this case, 250 is an approximation. Approximation involves rounding off and truncating.

Rounding Off

In rounding off, the place value to which a number is to be rounded must be stated. The digit occupying the next lower place value is considered. The number is rounded up if the digit is equal or greater than 5 and rounded down if it is less than 5.

Example 10

Round off 395.184 to:

(a) the nearest hundreds.
(b) four significant figures.
(c) the nearest whole number.
(d) two decimal places.

Solution

(a) 400 (b) 395.2 (c) 395 (d) 395.18

Truncation

Truncation means cutting off numbers to the given decimal places or significant figures, ignoring the rest.

Example 11

Truncate 3.246 to 2 decimal places.

Solution

3.246 is 3.24 truncated to 2 d.p.

Example 12

Truncate 561.7 to:

(a) 3 significant figures.
(b) 2 significant figures.

Solution

(a) 561.7 is 561 truncated to 3 s.f.

(b) 561.7 is 560 truncated to 2 s.f.

Estimation

Estimation involves rounding off numbers in order to carry out a calculation faster to get an approximate answer. This acts as a useful check on the actual answer.

Example 13

Estimate the answer to $\frac{152 \times 269}{32}$

Solution

The answer should be close to $\frac{150 \times 270}{30} = 1\,350$

The exact answer is 1277.75. 1277.75 written to 2 significant figures is 1 300, which is close to the estimated answer.

Exercise 2.2

1. Round off the following numbers to the nearest:
   (a) thousands.
   (b) hundreds.
   (c) tens.
   (i) 79 546 (ii) 65 324 (iii) 67 238
   (iv) 4 031 (v) 379 876 (vi) 13 188
2. Round off the following to the nearest:
   (a) 1 decimal place.
   (b) 2 significant figures.
   (i) 7.583 (ii) 8.00695 (iii) 17.366
   (iv) 0.0898 (v) 4.2039 (vi) 0.00283
3. Round off the following to the nearest:
   (a) tenths.
   (b) hundredths.
   (c) thousandths.
   (i) 67.246 (ii) 1.2578 (iii) 0.03791
   (iv) 0.0369 (v) 32.9263 (vi) 9.5396
4. Truncate the following numbers to the accuracy given in brackets:
   (a) 709.398 (1 d.p.) (b) 3.321 (2 s.f.) (c) 16 988 (4 s.f.)
   (d) 56.615 (2 s.f.) (e) 0.06291 (3 d.p.) (f) 56.605 (2 d.p.)

5. Estimate the answers to:

   (a) (i) $369 + 51$ (ii) $3\,489 + 6\,699$ (iii) $3\frac{1}{8} + 7\frac{8}{9}$

   (b) (i) $1\frac{3}{4} - \frac{1}{2}$ (ii) $459.4 - 137.4$ (iii) $4572.6 - 178.2 + 79.8$

   (c) (i) 3.3×0.9 (ii) 79.1×151 (iv) $2\,367 \times 19\,879$

   (d) (i) $25.4 \div 2.4$ (ii) $1\frac{7}{8} \div \frac{3}{4}$ (iii) $2\,492 \div 32$

   (e) (i) $\frac{125 \times 11.2}{9.8}$ (ii) $\frac{396.4 \times 7.15}{4.2}$ (iii) $\frac{523.4 \div 13.3}{8.2 \times 5.13}$

6. A motorist travelled for $2\frac{2}{3}$ hours and covered a distance of 569 km. Estimate his speed.
7. Estimate the time which a bus travelling at 79.67 km/h will take to cover a distance of 3 598 km.

2.3: Accuracy and Errors

In everyday life, we encounter measurements such as:

(i) the capacity of a water tank.
(ii) the length of a rope.
(iii) the mass of a baby.
(iv) room temperature, and so on.

Such quantities are expected to have actual values. However, when they are measured, there is always a small difference between the actual value and the measured value no matter how accurate the measuring instrument is. Hence the measured values are approximations (estimates) of the actual values. The difference between the actual value and the measured value is referred to as the error.

Accuracy of Instruments

An ordinary ruler can be used to measure length up to the nearest millimetre. When a length is given as 3.6 cm, it means it is nearer 3.6 cm, than 3.5 or 3.7 cm. The measurement lies between 3.55 cm and 3.65 cm. Therefore, 3.55 cm is the lower limit while 3.65 cm is the upper limit.

A micrometer screw gauge or vernier calippers can measure length up to the nearest thousandths of a centimetre. When a length is measured as 2.348 cm, it means it is nearer 2.348 cm than 2.347 cm or 2.349 cm. The measurement lies between 2.3475 cm and 2.3485 cm, where the two are lower and upper limits respectively. Thus, a micrometer screw gauge or vernier calippers measures lengths to a higher degree of accuracy than a ruler. Similarly, a digital stop watch will measure time to a higher degree of accuracy than a wall clock, an electronic balance gives a higher level of accuracy in measuring masses than an ordinary balance. Can you think of more accurate instruments that measure capacity, temperature and electric current?

Absolute Error

When a measurement is stated as 3.6 cm to the nearest millimetre, it lies between 3.55 cm and 3.65 cm. The least unit of measurement in this case is the millilitre, or 0.1 cm. The greatest possible error is $3.55 - 3.6 = -0.05$ or $3.65 - 3.6 = +0.05$.
Since we are only interested in the size of the possible error, we ignore the sign and say that the absolute error is 0.05. Thus, $|-0.05| = |+0.05| = 0.05$.
If l centimetres is the exact length, then $3.55 \leq l \leq 3.65$ or $l = 3.6 \pm 0.05$.

When a measurement is stated as 2.348 cm to the nearest thousandths of a centimetre (0.001), then the absolute error is $\frac{1}{2} \times 0.001 = 0.0005$.

Generally, the absolute error of a stated measurement is half of the least unit of measurement used, unless otherwise stated.

Relative Error

An error of 0.5 kg in measuring the mass of a cock is much more serious than an error of 0.5 kg in measuring the mass of a bull. If the actual mass of a cock is 2 kg and that a bull is 200 kg, then the relative error with respect to the cock is;

$\frac{0.5 \text{ kg}}{2 \text{ kg}} = 0.25$, while that with respect to the bull is;

$$\frac{0.5 \text{ kg}}{200 \text{ kg}} = 0.0025$$

$$\text{Relative error} = \frac{\text{absolute error}}{\text{actual measurement}}$$

Percentage Error

$$\begin{aligned}\text{Percentage error} &= \text{relative error} \times 100\ \%\\ &= \frac{\text{absolute error}}{\text{actual measurement}} \times 100\ \%\end{aligned}$$

In the examples above:

(i) for the cock, percentage error $= \frac{0.5 \text{ kg}}{2 \text{ kg}} \times 100\ \%$

$= 25\ \%$

(ii) for the bull, percentage error $= \frac{0.5 \text{ kg}}{200 \text{ kg}} \times 100\ \%$

$= 0.25\ \%$

A quantity which has 25 % error is less reliable than that which is 0.25% in error.

Example 14

The capacity of a gourd is stated as 2 000 millilitres to the nearest 10 millilitres.

Find: (a) the limits within which the capacity lies.

(b) the percentage error.

Solution

(a) Absolute error is $\frac{1}{2}(10) = 5$ ml

Lowest possible capacity is 2 000 ml – ~~5ml = 1 995~~ ml.

Highest possible capacity is 2 000 ml + 5 ml = 2 005 ml.

(b) Percentage error $= \frac{\text{absolute error}}{\text{actual measurement}} \times 100\ \%$

$= \frac{5 \text{ ml}}{2\,000 \text{ ml}} \times 100\ \%$

$= 0.25\ \%$

Work through this example by taking the capacity of the gourd as 2 litres to the nearest 0.01 litres.

Example 15

The thickness of a coin is 0.20 cm.

(a) What is the percentage error?

(b) What would be the percentage error if its thickness was stated as 0.2 cm?

Solution

(a) The smallest unit of measurement is 0.01.

Absolute error $= \frac{1}{2} \times 0.01$

$= 0.005$

Percentage error $= \frac{0.005}{0.20} \times 100\ \%$

$= 2.5\ \%$

(b) The smallest unit of measurement is 0.1.

Absolute error $= \frac{1}{2} \times 0.1 = 0.05$ cm

Percentage error $= \frac{0.05}{0.2} \times 100\%$

$= 25\%$

Note:
The zero in the second decimal place in (a) is significant and tells us that measurement is accurate to 2 decimal places while that in (b) is accurate to one decimal place.

Round Off and Truncation Errors

Rounding Off Error
If a number is rounded off to a desired number of decimal places or significant figures, an error is introduced in so far as the new number is different from the original one. Such an error is called a **round off error**.

For instance, when a recurring decimal $1.\dot{6}$ is rounded to 2 significant figures, it becomes 1.7. The round off error is;

$$1.7 - 1.6 = \tfrac{17}{10} - \tfrac{5}{3}$$
$$= \tfrac{1}{30}$$

Note:
1.6 converted to a fraction is $\tfrac{5}{3}$.

Truncation Error
The error introduced due to truncation is called a **truncation error**. In the case of 1.6 truncated to 2 s.f., the truncation error is; $|1.6 - 1.\dot{6}| = |1\tfrac{6}{10} - 1\tfrac{2}{3}|$

$$= \tfrac{1}{15}$$

Exercise 2.3

1. Find the limits within which the measurements below lie:
   (a) 26 cm (b) 26.0 cm (c) 26.07 cm
2. For each of the following, state the upper and lower limits of the measurements and the absolute error:
   (a) 1.025 kg (b) 2.50 m (c) 158 km
   (d) 12.10 s (e) 12.1 s (f) 12 s
3. State the greatest possible error in each of the following measurements:
   (a) 10 s (b) 11.0 s (c) 3.14 cm (d) 98.4 °F (e) 40 min
4. The temperature of a body is measured and recorded as 29.5 °C. Find:
   (a) the absolute error.
   (b) the relative error.
   (c) the percentage error.
5. An angle was measured to the nearest degree as 40^0. Calculate:

(a) the maximum possible value of the stated angle.
(b) the minimum possible value of the stated angle.
(c) the percentage error.

6. The best time for 100 m dash on a school sports day is given as 10.7 s. State within which limits the actual time lies.
7. Find the absolute error, the relative error and percentage error in each of the following measurements:
   (a) 45.38 s (b) 0.45 g (c) 7.62 *l*
   (d) 12.004 cm (e) 2 505 tonnes
8. The number 2.5 has been rounded off to one decimal place. Calculate:
   (a) the range within which the exact value lies.
   (b) the relative error.
9. The thickness of a book was measured to the nearest 0.02 cm as 8.0 cm. What is the percentage error?
10. What is the percentage error in using:
    (a) 0.67 (b) 0.66

    as an estimate of $\frac{2}{3}$?
11. For each of the following numbers:
    (a) round off to 2 significant figures and calculate the percentage error.
    (b) truncate to 2 significant figures and calculate the percentage error.
    (i) 0.327 (ii) 4.05 (iii) 2.718
    (iv) 3.14 (v) 10.36 (vi) 395
12. The following numbers are rounded to the degree of accuracy in brackets. Find the upper and lower bounds:
    (a) −186.00 (2 d.p.)
    (b) 8 000 (to the nearest 100)
    (c) −43 000 (to the nearest thousand)
    (d) 694.0 (to the nearest 0.1)

2.4: Propagation of Errors

When approximate values are used in addition, subtraction, multiplication and division, the resultant values are approximations of the actual value. This is illustrated in examples below.

Addition and Subtraction

Example 16

What is the error in the sum of 4.5 cm and 6.1 cm, if each represents a measurement?

Solution

The limits within which the measurements lie are 4.45, i.e., 4.55 or 4.5 ± 0.005 and 6.05 to 6.15, i.e, 6.1 ± 0.05.

The maximum possible sum is 4.55 + 6.15 = 10.7 cm.

The minimum possible sum is 4.45 + 6.05 = 10.5 cm.

The working sum is 4.5 + 6.1 = 10.6

$$\begin{aligned}\text{The absolute error} &= \text{maximum sum} - \text{working sum}\\ &= |10.7 - 10.6|\\ &= 0.10\end{aligned}$$

Example 17

What is the error in the difference between the measurements 0.72 g and 0.31 g?

Solution

The measurements lie within 0.72 ± 0.005 and 0.31 ± 0.005 respectively. The maximum possible difference will be obtained if we subtract the minimum value of the second measurement from the maximum value of the first, i.e.; 0.725 – 0.305 cm see figure 2.2.

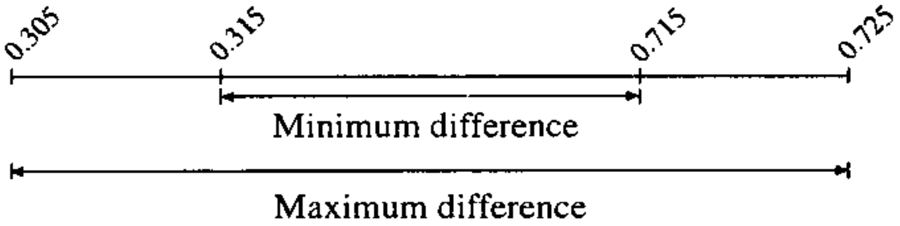


Fig. 2.2

The minimum possible difference is 0.715 – 0.315 = 0.400. The working difference is 0.72 – 0.31 = 0.41, which has an absolute error of |0.420 – 0.41| or |0.400 – 0.41| = 0.10. Since our working difference is 0.41, we give the absolute error as 0.01 (to 2 s.f.).

Note:

In both addition and subtraction, the absolute error in the answer is equal to the sum of the absolute errors in the original measurements.

Find the relative errors in examples 16 and 17.

Multiplication

Example 18

A rectangular card measures 5.3 cm by 2.5 cm. Find:

(a) the absolute error in the area of the card.

(b) the relative error in the area of the card.

Solution

(a) The length lies within the limits 5.3 ± 0.05 cm.
The breadth lies within the limits 2.5 ± 0.05 cm.
The maximum possible area is $2.55 \times 5.35 = 13.6425$ cm^2.
The minimum possible area is $2.45 \times 5.25 = 12.8625$ cm^2.
The working area is $5.3 \times 2.5 = 13.25$ cm^2.
Maximum area – working area = $13.6425 - 13.25 = 0.3925$.
Working area – minimum area = $13.25 - 12.8625 = 0.3875$.
Note that the working area (product) is not midway between the lower and upper limits.
We may take the absolute error as the average of the two.

$$\text{Thus, absolute error} = \frac{0.3925 + 0.3875}{2}$$
$$= 0.3900$$

The error in the area is therefore 0.39. The same can also be found by taking half the interval between the maximum area and the minimum area, i.e.;

$$\tfrac{1}{2}(13.6425 - 12.8625) = 0.39$$

(b) The relative error in the area is;

$$\frac{0.39}{13.25} = 0.029 \text{ (to 2 s.f.)}$$

Alternatively;

$$\text{Relative error in length is } \frac{0.05}{5.3} = \frac{5}{530}$$
$$= 0.0094$$

$$\text{Relative error in width is } \frac{0.05}{2.5} = \frac{5}{250}$$
$$= 0.02$$

$$\text{Sum of the relative errors in length and width} = 0.0094 + 0.02$$
$$= 0.0294$$
$$= 0.029 \text{ (to 2 s.f.)}$$

Generally, the relative error in a product is approximately equal to the sum of the relative errors in the individual values.

Division

Example 19

Given 8.6 cm ÷ 3.4 cm, find :

(a) the absolute error in the quotient.

(b) the relative error in the quotient .

Solution

(a) 8.6 cm has limits 8.55 cm and 8.65 cm. 3.4 has limits 3.35 cm and 3.45 cm. The maximum possible quotient will be given by the maximum possible value of the numerator and the smallest possible value of the denominator, i.e.,

$\frac{8.65}{3.35} = 2.58$ (to 3 s.f.)

The minimum possible quotient will be given by the minimum possible value of the numerator and the biggest possible value of the denominator, i.e.;

$\frac{8.55}{3.45} = 2.48$ (to 3 s.f.)

The working quotient is; $\frac{8.6}{3.4} = 2.53$ (to 3 s.f.)

The absolute error in the quotient is;

$$\begin{aligned}\frac{2.58 - 2.48}{2} &= \tfrac{1}{2} \times 0.10\\ &= 0.050 \text{ (to 2 s.f.)}\end{aligned}$$

(b) Relative error in the working quotient is;

$$\begin{aligned}\frac{0.05}{2.53} &= \frac{5}{253}\\ &= 0.0197\\ &= 0.020 \text{ (to 2 s.f.)}\end{aligned}$$

Alternatively;

Relative error in the numerator is $\frac{0.05}{8.6} = 0.00581$

Relative error in the denominator is $\frac{0.05}{3.4} = 0.0147$

Sum of the relative errors in the numerator and denominator is

$$\begin{aligned}0.00581 + 0.0147 &= 0.02051\\ &= 0.021 \text{ (to 2 s.f.)}\end{aligned}$$

Again, we see that the relative error in the quotient is approximately equal to the sum of the relative errors in the numerator and denominator.

Exercise 2.4

1. Find the minimum and maximum possible sums of the following measurements:
   (a) 10.34 cm and 0.84 cm (b) 8.63 *l* and 3.52 *l*
   (c) 450 m, 32.6 m and 0.39 m
2. (a) The sides of a triangle are 22.5 cm, 23 cm, 25.7 cm long. Within which limits does its perimeter lie?

(b) A rectangular field measures 40 m by 32.5 m. Within which limits does it perimeter lie?

3. The first and second athletes in 9 200 m race were timed at 21.1 s and 21.3 s respectively. Calculate the minimum and maximum possible differences between their times.

4. A pipe 3.0 m long was cut into three pieces. The first and second pieces measured 1.3 m and 0.94 m respectively. Find the limits within which the length of the third piece lies.

5. Find the minimum possible perimeter of a regular pentagon whose side is 15.0 cm.

6. Find the lower and upper limits of the differences between:
   (a) 26.0 cm and 14.2 cm (b) 0.08 km and 0.093 km
   (c) 3.48 g and 1.29 g (d) 3.45 *l* and 1.25 *l*

7. Ten washers, each 3.7 mm in thickness, are piled one on top of the other. Find the maximum possible error in the calculation of the height of the pile.

8. The dimensions of a box of matches are stated as 5.2 cm × 3.8 cm × 1.5 cm. Within which limits does its:
   (a) volume, and,
   (b) surface area lie?

9. Temperatures, to the nearest 0.1 ^{0}C, are stated as $a = 2.7$ ^{0}C, $b = 3.4$ ^{0}C, $c = 9.8$ ^{0}C and $d = 3.0$ ^{0}C. Find the relative errors in each of the following expressions:
   (a) $a + b$ (b) $d - c$ (c) $a - b + c$ (d) ac
   (e) abc (f) $a(d + c)$ (g) $bc - ad$ (h) $\frac{a}{c}$
   (i) $\frac{a + d}{c}$ (j) $\frac{d}{ac}$ (k) $\frac{a + b}{c + d}$ (l) $\frac{b}{c} - \frac{c}{d}$

10. The radius and height of a cylindrical mug are 4.5 cm and 8.2 cm respectively. Find the limits within which its capacity lies (leave your answer in terms of π).

11. Find the limits within which the area of a parallelogram whose base is 8 cm and height is 5 cm lies. Hence, find the relative error in the area.

12. Six girls have masses given to the nearest 10 kg, as 40 kg, 50 kg, 60 kg, 60 kg, 70 kg and 80 kg. Calculate:
    (a) the maximum possible sum of their mass.
    (b) the minimum possible sum of their mass.

13. The dimensions of a rectangle are 10 cm and 15 cm. If there is an error of 5 % in each of the measurements, find the maximum and least possible area of the rectangle.

14. Two towns on the earth surface are 17 000 km apart to the nearest thousands. The average speed of a supersonic plane to the nearest five hundred is 3 500 km/h. Calculate the maximum possible time that the plane will take to cover the distance between the towns.
15. Find the percentage error in the calculation of the volume of a sphere radius 4.9 cm.
16. A triangle ABC is right angled at B. AB = 5 cm and BC = 12 cm. Find the relative error in the calculation of its area using each of the formulae:

    (a) $A = \frac{1}{2}bh$

    (b) $A = \frac{1}{2}ab\sin c$

    (c) $A = \sqrt{s(s-a)(s-b)(s-c)}$

Chapter Three

TRIGONOMETRY (II)

Introduction

You have learnt to find trigonometric ratios of angles between 0^0 and 90^0 (inclusive) from the right-angled triangle and mathematical tables.

In this section, we shall consider trigonometric ratios:

(i) in terms of co-ordinates of points for all angles (positive and negative).

(ii) of angles greater than 90^0 and negative angles using mathematical tables.

(iii) of all angles (positive and negative) using the calculator.

3.1: The Unit Circle

Figure 3.1 is a circle of unit radius and centre O(0,0). We call this circle the unit circle.

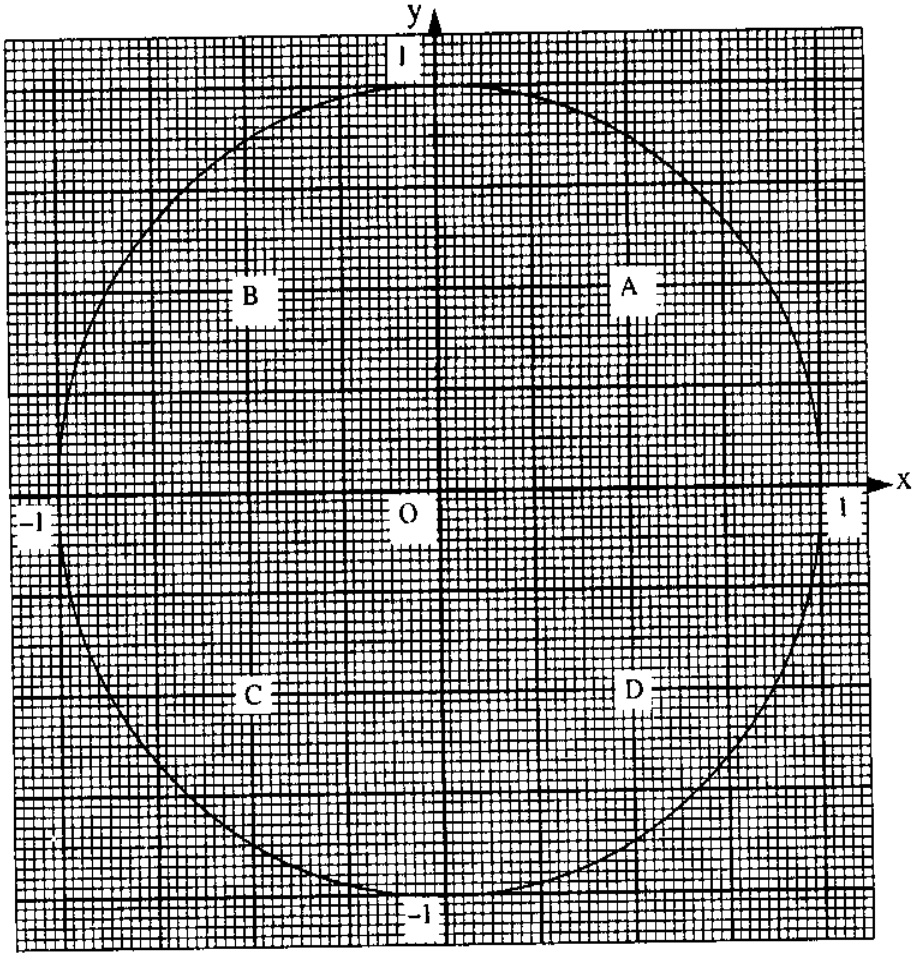


Fig. 3.1

The sectors of the circle in figure 3.1 labelled A, B, C and D are called first, second, third and fourth quadrants respectively. An angle measured anticlockwise from OX (positive direction of x-axis) is positive. An angle measured clockwise from OX is negative.

Figure 3.2 shows angles of 120⁰ and –50⁰ marked on the unit circle. They are in the second and fourth quadrants respectively.

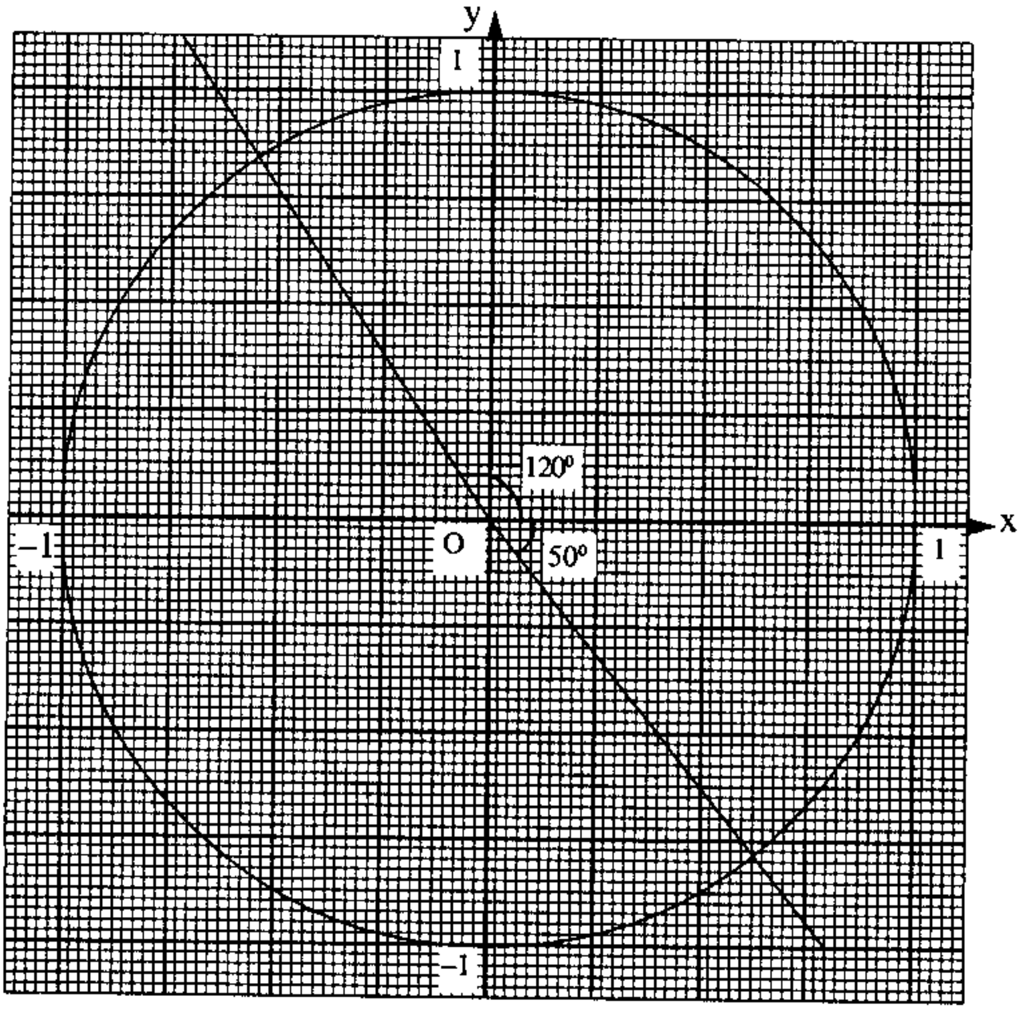


Fig. 3.2

In which quadrants are the angles 30⁰, 140⁰, 240⁰, 330⁰, –70⁰ and –120⁰?

Figure 3.3 is a unit circle and angle PON is 30⁰. What are the values of x and y at point P?

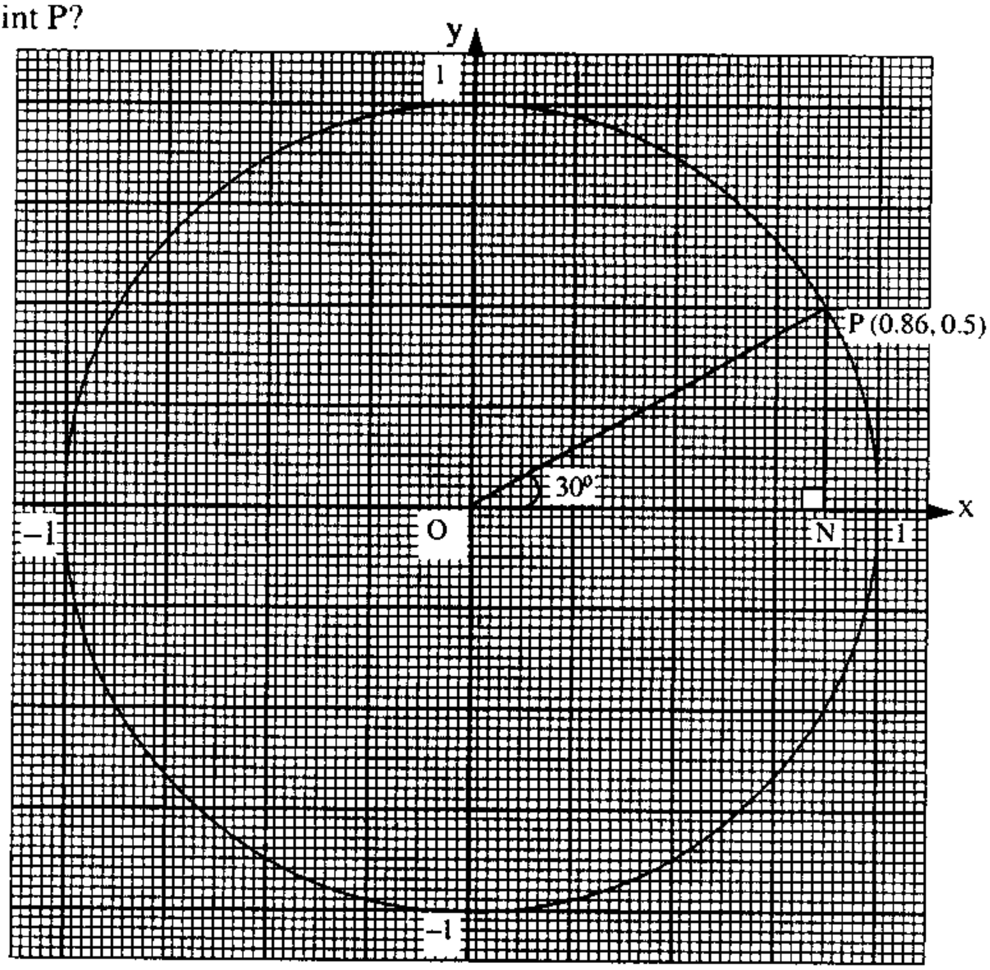


Fig. 3.3

You realise that ΔONP is right-angled at N. Therefore;

$$\sin 30^0 = \frac{NP}{OP}$$

$$= \frac{0.5}{1}$$

= 0.5, which is the y co-ordinate of P.

$$\cos 30^0 = \frac{ON}{OP}$$

$$= \frac{0.86}{1}$$

= 0.86, which is the X co-ordinate of P.

$$\tan 30^0 = \frac{NP}{ON}$$

$$= \frac{0.5}{0.86}$$

$$= \frac{\text{y co-ordinate}}{\text{x co-ordinate}} \text{ on the unit circle}$$

In general, on a unit circle (see figure 3.4):

(i) $\cos\theta$ = x co-ordinate of P.

(ii) $\sin\theta$ = y co-ordinate of P.

(iii) $\tan\theta = \frac{\text{y co-ordinate P}}{\text{x co-ordinate P}} = \frac{\sin\theta}{\cos\theta}$

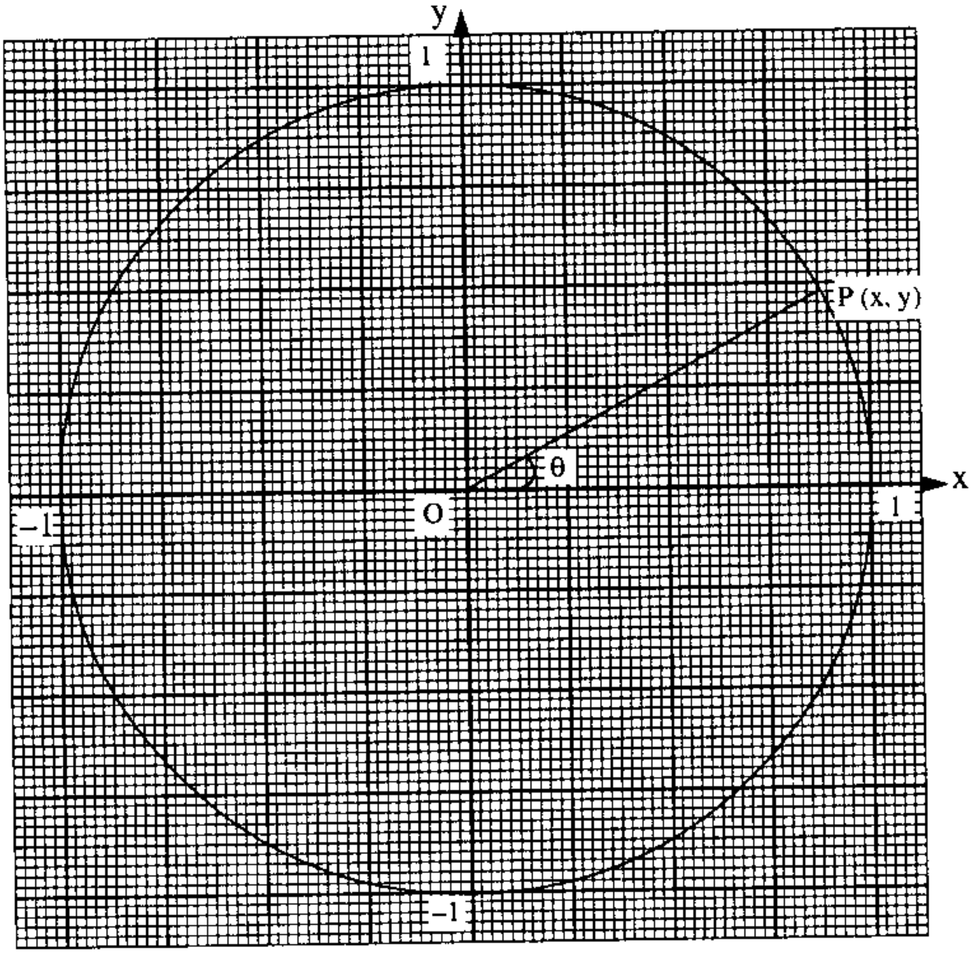


Fig. 3.4

Use the unit circle to find the trigonometric ratios of angles 60°, 45° and 75°.

3.2: Trigonometric Ratios of Angles Greater Than 90°

We can also find the trigonometric ratios of angles greater than 90° using a unit circle. As P moves round the circle, ∠POX varies.

For example, in figure 3.5, P_2, P_3 and P_4 are points on a unit circle such that;
$\angle P_2OX = 150^0$, $\angle P_3OX = 210^0$ and $\angle P_4OX = 330^0$.

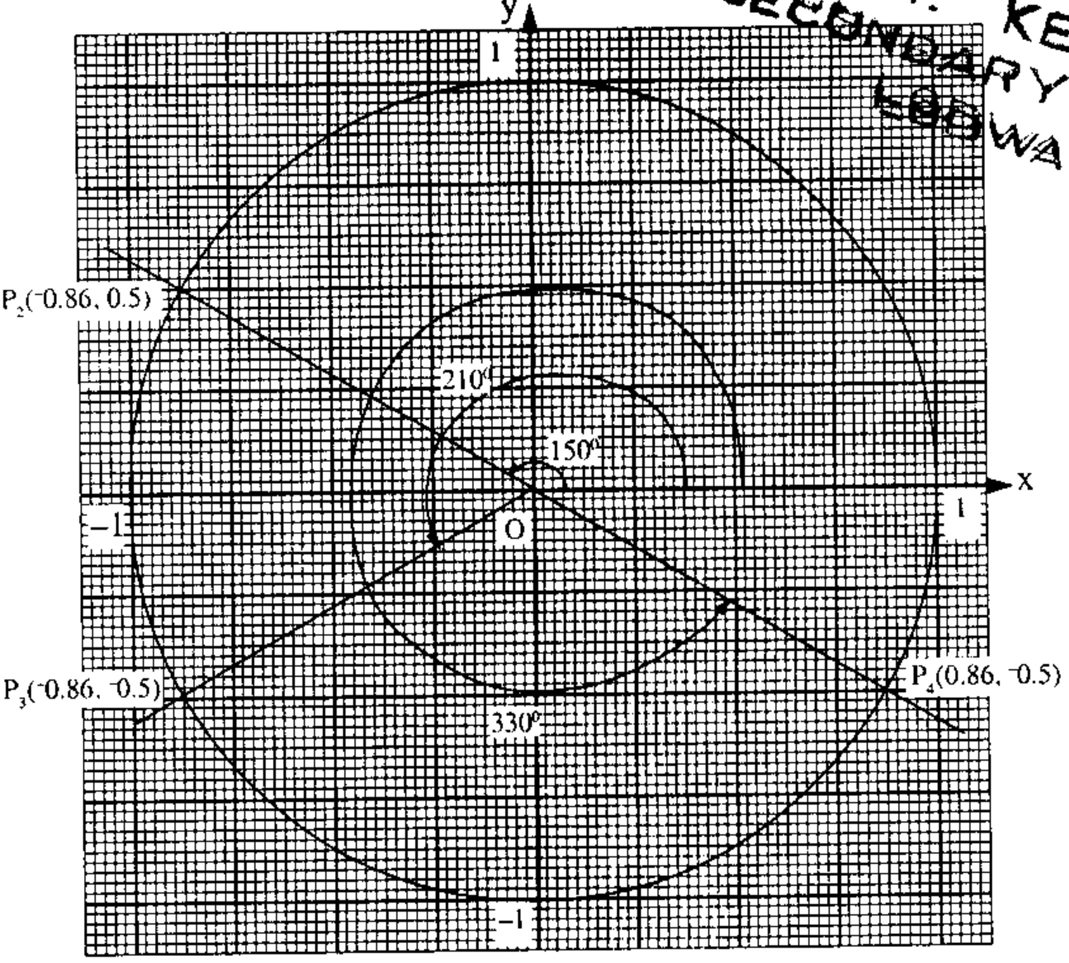


Fig. 3.5

The trigonometric values of these angles can be found from the graph as follows:

(i) Reading the values of x and y co-ordinates of point P_2, we get;
$\cos 150^0 = -0.86$
$\sin 150^0 = 0.5$

$$\therefore \tan 150^0 = \frac{0.5}{-0.86} = -0.58 \text{ (2 s.f.)}$$

Use a unit circle to find the values of cosine, sine and tangent of 120^0, 135^0 and 105^0.

(ii) Reading the values of x and y co-ordinates of point P_3, we get;
$\cos 210^0 = -0.86$
$\sin 210^0 = -0.5$

$\tan 210^0 = \frac{-0.5}{-0.86} = 0.58$ (2 s.f.)

Use a unit circle to find the values of cosine, sine and tangent of 240^0, 225^0 and 255^0.

(iii) Reading the values of x and y coordinates of point P_4, we get;

$\cos 330^0 = 0.86$

$\sin\ 330^0 = -0.5$

$\tan\ 330^0 = \frac{-0.5}{0.86} = -0.58$ (2 s.f.)

You notice that:

(i) all trigonometric ratios of angles in the first quadrant are positive.

(ii) only values of sines of angles in the second quadrant are positive.

(iii) only values of tangents of angles in the third quadrant are positive.

(iv) only values of cosines of angles in the fourth quadrant are positive.

We can express trigonometric ratios of angles greater than 90^0 in terms of trigonometric ratios of acute angles. For example;

(i) From the unit circle in figure 3.5, we can read the values of cosines of 30^0, 150^0, 210^0 and 330^0 as given in the following table:

Table 3.1

0	30^0	150^0	210^0	330^0
Cos θ	0.86	−0.86	−0.86	0.86

You notice that $\cos 30^0 = -\cos 150^0 = -\cos 210^0 = +\cos 330^0$.

This can be rewritten as;

$$\begin{aligned}\cos 30^0 &= -\cos(180^0 - 30^0)\\ &= -\cos(180^0 + 30^0)\\ &= +\cos(360^0 - 30^0)\end{aligned}$$

Using the unit circle, find the values of cosines of the following angles; 53^0, 127^0, 233^0 and 307^0. What is the relationship between the cosine of the acute angle 53^0 and the cosines of each of the remaining angles?

(ii) From the unit circle in figure 3.5, we can read the values of sines of 45^0, 135^0, 225^0 and 315^0 as shown in table 3.2.

Table 3.2

θ	45^0	135^0	225^0	315^0
sin θ	0.7	0.7	−0.7	−0.7

You notice that $\sin 45^0 = \sin 135^0 = -\sin 225^0 = -\sin 315^0$

This can be rewritten as;

$\sin 45^0 = \sin(180^0 - 45^0)$

$= -\sin(180^0 + 45^0)$
$= -\sin(360^0 - 45^0)$

Using a unit circle, find the values of the sines of the following angles 23^0, 157^0, 203^0 and 337^0. What is the relationship between the sine of the acute angle 23^0 and the sines of each of the remaining angles?

(iii) With the help of a unit circle, copy and complete the following table;

Table 3.3

θ	30^0	150^0	210^0	330^0
$\sin\theta$	0.5			
$\cos\theta$				
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$				

You should realise that $\tan 30^0 = -\tan 150^0 = +\tan 210^0 = -\tan 330^0$

Thus, $\tan 30^0 = -\tan(180^0 - 30^0)$
$= +\tan(180^0 + 30^0)$
$= -\tan(360^0 - 30^0)$

From the activities in (i) to (iii), you should have noticed that for an acute angle θ;

(i) $\sin\theta = +\sin(180^0 - \theta) = -\sin(180^0 + \theta) = -\sin(360^0 - \theta)$
(ii) $\cos\theta = -\cos(180^0 - \theta) = -\cos(180^0 + \theta) = +\cos(360^0 - \theta)$
(iii) $\tan\theta = -\tan(180^0 - \theta) = +\tan(180^0 + \theta) = -\tan(360^0 - \theta)$

Exercise 3.1

Use the unit circle in this exercise.

1. (a) Find the value of $\sin 20^0$. Hence, state the value of $\sin 160^0$, $\sin 200^0$ and $\sin 340^0$.
   (b) Find the value of $\sin 224^0$. Hence, state the value of $\sin 44^0$, $\sin 136^0$ and $\sin 316^0$.
2. (a) Find the value of $\cos 25^0$. Hence, state the value of $\cos 155^0$, $\cos 205^0$ and $\cos 335^0$.
   (b) Find the value of $\cos 107^0$. Hence, state the value of $\cos 73^0$, $\cos 253^0$ and $\cos 287^0$.
3. (a) Find the value of $\tan 45^0$. Hence, state the value of $\tan 135^0$, $\tan 225^0$ and $\tan 315^0$.
   (b) Find the value of $\tan 145^0$. Hence, state the value of $\tan 215^0$, $\tan 35^0$ and $\tan 325^0$.
   (c) Find the value of $\tan 302^0$. Hence, state the value of $\tan 122^0$, $\tan 58^0$ and $\tan 238^0$.

4. Write down the values of the cosine and sine of each of the following angles: 0^0, 90^0, 180^0, 270^0 and 360^0.
5. Find the values of the tangents of 0^0, 45^0, 135^0, 180^0, 225^0, 315^0 and 360^0. What can you say about tan 90^0 and tan 270^0?
6. Given that $\cos\theta = 0.6$ and θ is an acute angle, Find:
   (a) $\sin\theta$ (b) $\tan\theta$
7. If sin = 0.8, find all possible values of θ between 0^0 and 360^0.
8. What can you say about angle θ in the range $O^0 \le \theta \le 360^0$ if:
   (a) $\cos\theta$ is negative while $\tan\theta$ is positive?
   (b) $\sin\theta$ is positive while $\tan\theta$ is negative?
   (c) $\cos\theta$ is positive while $\sin\theta$ is negative?

3.3: Trigonometric Ratios of Negative Angles

Trigonometric ratios of negative angles can also be found by using a unit circle. Consider the use of a unit circle to find the sine, cosine and tangent of -60^0. The angle -60^0 is measured clockwise from the x-axis, as in figure 3.6.

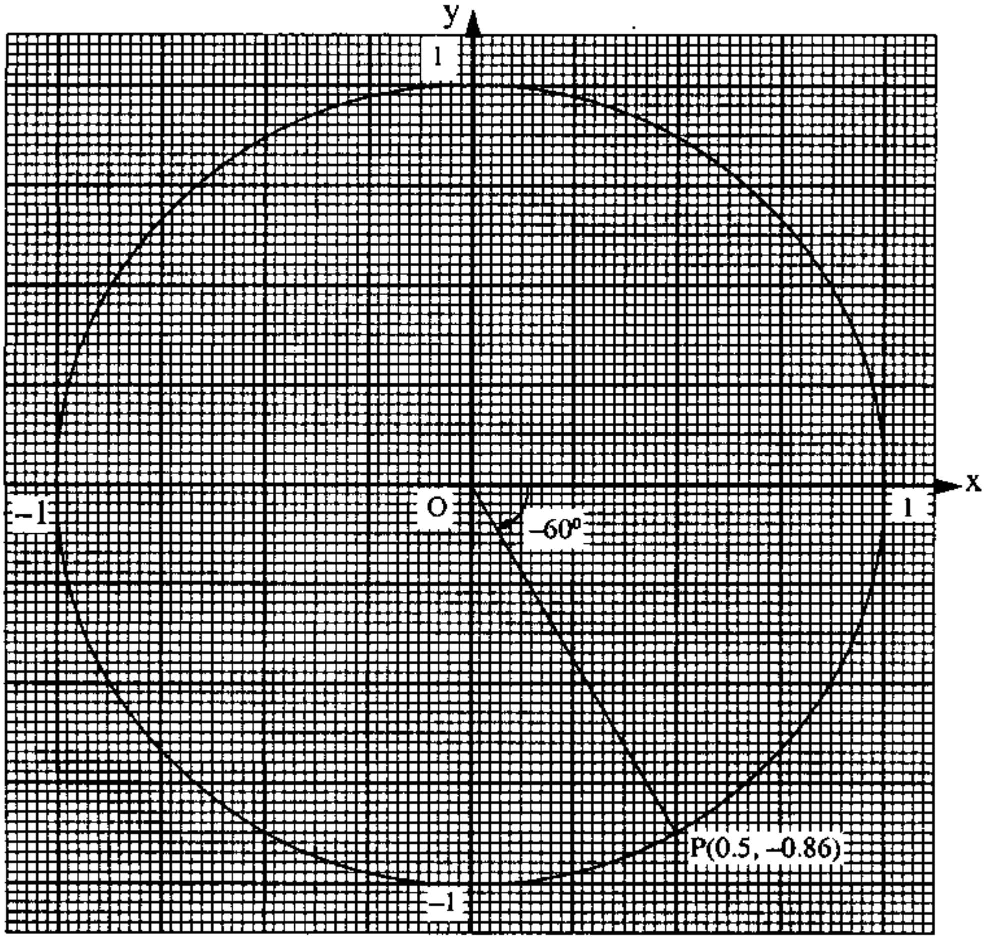


Fig. 3.6

Reading the values of x and y co-ordinates of point P we get;
sin $(-60^0) = 0.86$
cos $(-60^0) = 0.5$

$\therefore$ tan $(-60^0) = \frac{-0.86}{0.5} = -1.7$ (2 s.f.)

Use a unit circle to find the values of sine, cosine and tangents of -45^0, -135^0, -150^0 and -210^0.

Using a unit circle, copy and complete the following table.

Table 3.4

θ	30^0	-30^0
sin θ	0.5	—
cos θ	—	—
tan θ	—	—

Write down the relationship between:
(i) sin 30^0 and sin (30^0)
(ii) cos 30^0 and cos (-30^0)
(iii) tan 30^0 and tan (30^0)

You notice that:
(i) $\sin(30^0) = -\sin 30^0$
(ii) $\cos(-30^0) = \cos 30^0$
(iii) $\tan(-30^0) = -\tan 30^0$

Generally, for any angle θ:
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

3.4: Trigonometric Ratios of Angles Greater than 360^0

(a) In figure 3.7, P is a point on a unit circle such that $\angle POX = 390^0$.

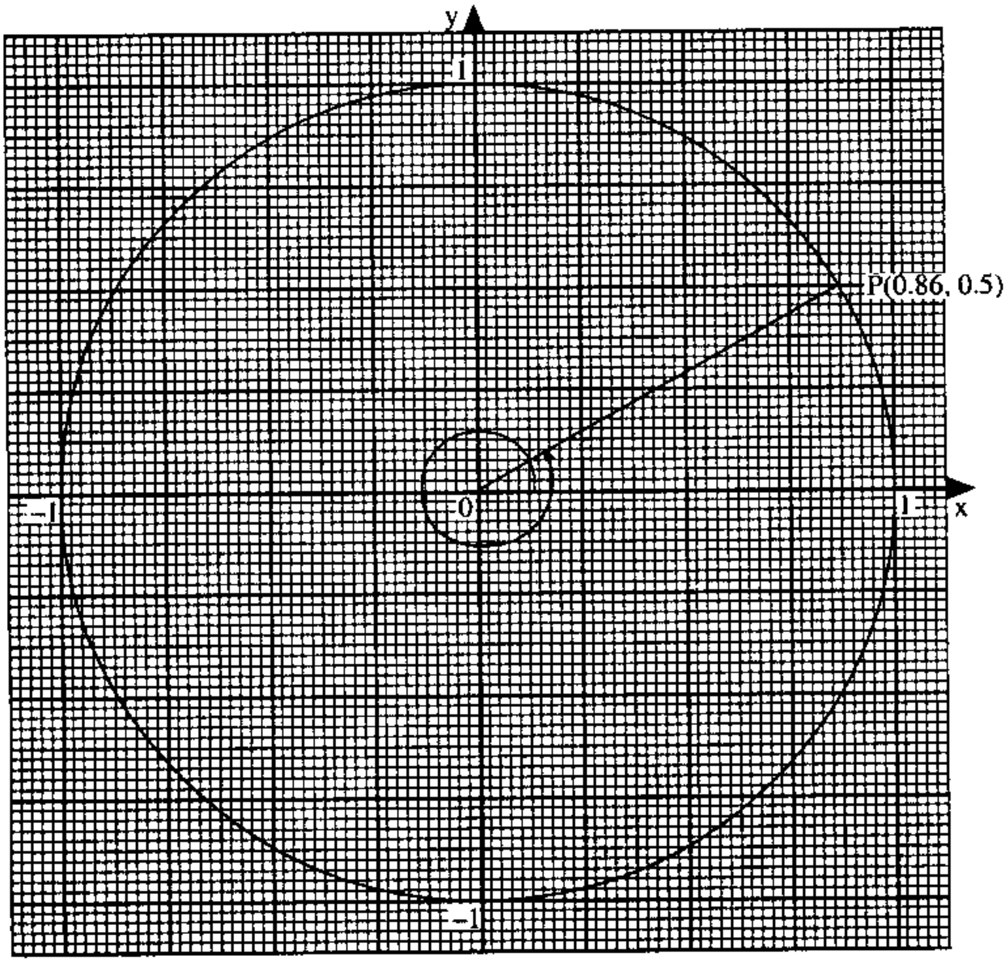


Fig. 3.7

Note:

390^0 is a complete rotation in a unit circle plus an overlap of 30^0 in the first quadrant, i.e., $390^0 = 360^0 + 30^0$. Reading the values of x and y co-ordinates of point P, we get;

$\cos 390^0 = 0.86$

$\sin 390^0 = 0.5$

$\tan 390^0 = \frac{0.5}{0.86} = 0.58$ (2 s.f.)

Use a unit circle to find the values of cosine, sine and tangent of 510^0, 570^0 and 690^0.

Generally, trigonometric ratios for any angle greater than 360^0 can be expressed in terms of the relevant acute angle.

For example;

(i) $\sin 390^0 = \sin (390^0 - 360^0) = \sin 30^0$

$\cos 390^0 = \cos (390^0 - 360^0) = \cos 30^0$

$\tan 390^0 = \tan (390^0 - 360^0) = \tan 30^0$

(ii) $\sin 480^0 = \sin (480^0 - 360^0) = \sin 120^0 = \sin 60^0$

$\cos 480^0 = \cos (480^0 - 360^0) = \cos 120^0 = -\cos 60^0$

$\tan 480^0 = \tan (480^0 - 360^0) = \tan 120^0 = -\tan 60^0$

(iii) $\sin 750^0 = \sin (750^0 - 360^0 \times 2) = \sin 30^0$

$\cos 750^0 = \cos (750^0 - 360^0 \times 2) = \cos 30^0$

$\tan 750^0 = \tan (750^0 - 360^0 \times 2) = \tan 30^0$

Find the sines, cosines and tangent of 500^0, 600^0, 840^0 and $1\,200^0$

Exercise 3.2

Use the unit circle in this exercise.

1. Show the following angles on the unit circle:
   (a) (i) -20^0 (ii) -160^0 (iii) -200^0 (iv) -340^0
   (b) (i) 400^0 (ii) 570^0 (iii) 910^0 (iv) $1\,100^0$
2. Express the following in terms of the sine of an acute angle:
   (a) $\sin -170^0$ (b) $\sin -310^0$ (c) $\sin 448^0$ (d) $\sin 750^0$
3. Express the following in terms of cosine of an acute angle:
   (a) $\cos -75^0$ (b) $\cos -380^0$ (c) $\cos 560^0$ (d) $\cos 910^0$
4. Write down the values of sine of each of the following angles:
   $-37^0, -143^0, -217^0, -323^0$.
5. Write down the value of the cosine of each of the following angles:
   $-60^0, -120^0, -240^0$ and -300^0
6. Write down the value of the tangent of each of the following angles:
   $-42^0, -138^0, -222^0$ and -318^0.
7. Find the values of the sine, cosine and tangent of the following angles:
   800^0, $1\,030^0$ and $1\,250^0$.
8. Find the values of $\sin (-1\,000^0)$, $\cos (-1\,000^0)$ and $\tan (-1\,000^0)$.

3.5: Use of Mathematical Tables

In the previous section, the unit circle was used to find trigonometric ratios of angles. We shall now consider the use of mathematical tables to find trigonometric ratios.

Example 1

Find the cosine of: (a) 127^0 (b) 240^0 (c) $307^0\ 24'$

Solution

(a) From the unit circle (figure 3.8), cos $127^0 = -\cos 53^0$
From the table of cosines, cos $53^0 = 0.6018$.
Therefore, cos $127^0 = -\cos 53^0 = -0.6018$.

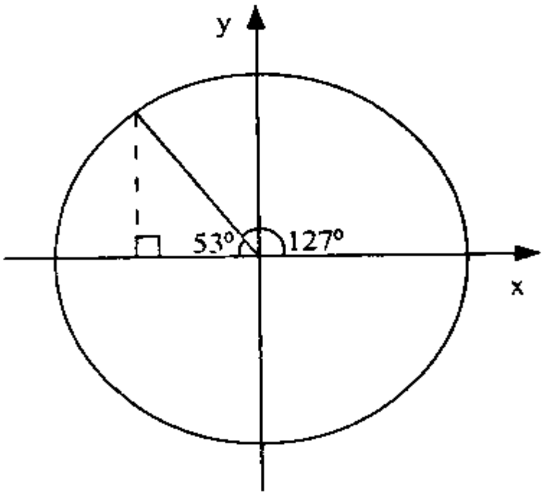


Fig. 3.8

(b) From the unit circle (figure 3.9), cos $240^0 = -\cos 60^0$
From the table of cosines, cos $60^0 = 0.5000$
Therefore, cos $240^0 = -\cos 60^0 = 0.5000$

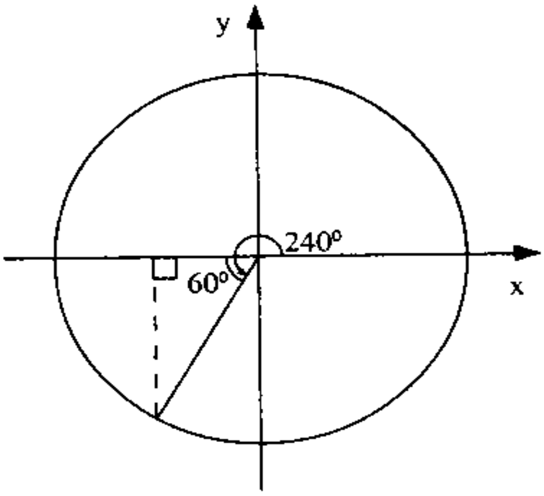


Fig. 3.9

(c) From the unit circle (figure 3.9), cos $307^0 24' = +\cos 52^0 36'$
From the table of cosines, cos $52^0 36' = 0.6074$
Therefore, cos $307^0 24' = 0.6074$.

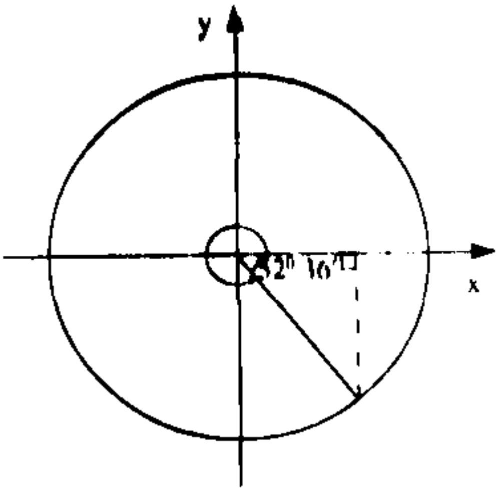


Fig 3.9

Example 2

Find the value of the sine of:

(a) 137.8^0 (b) 248^0 (c) 327^0

Solution

(a) From the unit circle (figure 3.10), sin 137.8^0 = sin 42.2^0
From the tables of sines, sine 42.2^0 = 0.6717
Therefore, sin 137.8^0 = 0.6717

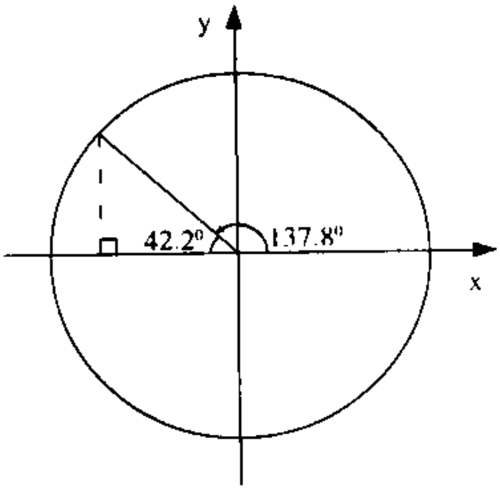


Fig. 3.10

(b) From the unit circle (figure 3.11, sin 248^0 = −sin 68^0
From tables of sines, sin 248^0 = −sin 68^0 = −0.9272

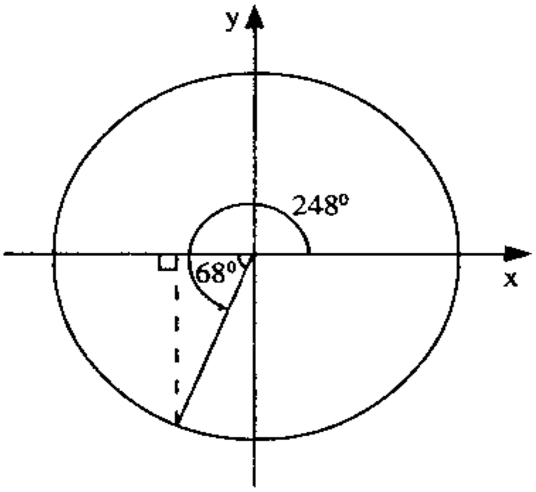


Fig. 3.11

(c) From the unit circle (figure 3.12, sin 327° = –sin 33°
From tables of sines, sin 33° = 0.5446
Therefore, sin 327° = – 0.5446

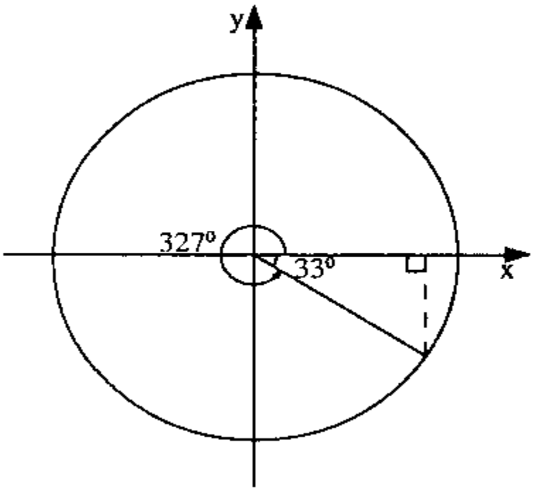


Fig. 3.12

From the tables, find the tangents of 72°, 159°, 234° and 290°.

Example 3

Find angles between 0° and 360° whose:
(a) sine is 0.70 71
(b) cosine is – 0.8221
(c) tangent is 1.511

Solution

(a) Sine is positive in the 1st and 2nd quadrants. From the table of sines, the acute angle whose sine is 0.7071 is 45^0. Therefore, the angle is 45^0 or 135^0.

(b) Cosine is negative in the 2nd and 3rd quadrants. From the table of cosines, the acute angle whose cosine is 0.8221 is 34.7^0. Therefore, the angle whose cosine is -0.8221 is 145.3^0 or 214.7^0.

(c) Tangent is positive in the 1st and 3rd quadrants. From the table of tangents, the acute angle whose tangent is 1.511 is 56.5^0.
Therefore, the angle whose tangent is 1.511 is 56.5^0 or 236.5^0.

Example 4

Solve the equation $\sin x = 0.8071$ for $0^0 < x \leq 360^0$.

Solution

Sine is positive in the 1st and 2nd quadrant. From the table of sines, the acute angle whose sine is 0.8071 is 53.81^0. Therefore, $x = 53.81^0$ or 126.19^0

Exercise 3.3

1. Express the following in terms of $\sin 40^0$:
   (a) $\sin 140^0$ (b) $\sin 220^0$ (c) $\sin 320^0$
   (d) $\sin(-400^0)$ (e) $\sin(-140^0)$
2. Express the following in terms of $\cos 50^0$:
   (a) $\cos 130^0$ (b) $\cos 230^0$ (c) $\cos 310^0$
   (d) $\cos(-50^0)$ (e) $\cos 410^0$
3. Express the following in terms of the tangent of an acute angle.
   (a) $\tan 132^0$ (b) $\tan 237^0$ (c) $\tan 327^0$
   (d) $\tan(-40^0)$ (e) $\tan 510^0$
4. Use mathematical tables to find the sine, cosine and tangent of each of the following:
   (a) 27.3^0 (b) $57^0 19'$ (c) $97^0 27'$ (d) 115.7^0
   (e) $131^0 36'$ (f) 164.7^0 (g) 177.2^0 (h) $193^0 46'$
   (i) 207.3^0 (j) 230^0 (k) 267.6^0 (l) $280^0 52'$
   (m) 300^0 (n) $329^0 30'$ (p) 347.23^0
5. (a) Using tables find the angles between 0^0 and 360^0 whose sine is:
   (i) 0.1124 (ii) 0.0971 (iii) -0.3729 (iv) 0.4032
   (v) -0.4924 (vi) -0.5643 (vii) -0.6478 (viii) 0.7124
   (ix) 0.7476 (x) -0.9304 (xi) 0.8888 (xii) -0.9056
   (xiii) 0.9497 (xiv) -0.9630 (xv) -1.000

(b) Repeat 5 (a) for the given values being cosines.

6. Using tables find the angles between -180^0 and 180^0 whose tangent is:
   (a) 0.0094 (b) 0.1987 (c) −0.2192 (d) 0.2541
   (e) 0.3721 (f) −0.4443 (g) −0.5786 (h) 0.7173
   (i) −0.9597 (j) −1.0951
7. Find all values of θ between 0^0 and 360^0 satisfying the equation;
   $\sin \theta = -0.8$
8. Express the following in terms of the sine of an acute angle:
   (a) $\cos 70^0$ (b) $\sin 300^0$ (c) $\cos 300^0$ (d) $\sin 200^0$
   (e) $\cos (-175^0)$
9. Calculate the area of triangle PQR in which PQ = 9 cm, PR = 6 cm and $\angle RPQ = 133^0$.
10. Find the area of the trapezium ABCD in which AB//DC, AB = 14 cm, CD = 7 cm, DA = 10 cm and $\angle ADC = 148^0$.

3.6: Use of Calculators

In this section, we shall use the calculator to solve problems involving trigonometry.

In finding trigonometric ratios (sine, cosine, tangent) of a given angle, the angle is keyed in followed by the ratio.

Example 5

Use a calculator to find:

(a) $\tan 30^0$ (b) $\tan 150^0$
(c) $\tan 330^0$ (d) $\tan 390^0$

Solution

(a) Step I Key in 30.
Step II Key in tan.
Screen displays 0.5773502

$$\therefore \tan 30^0 = 0.5773502$$
$$= 0.5774 \text{ (4 s.f.)}$$

(b) Step I Key in 150.
Step II Key in tan.
Screen displays −0.5773502

$$\therefore \tan 150^0 = -0.5773502$$
$$= -0.5774 \text{ (4 s.f.)}$$

(c) Step I Key in 330.
Step II Key in tan.

Screen displays –0.5773502

$\therefore$ tan 330^0 = –0.5773502

= –0.5774 (4 s.f.)

(d) Step I Key in 390.

Step II Key in tan.

Screen displays 0.5773502

$\therefore$ tan 390^0 = 0.5773502

= 0.5774 (4 s.f.)

The same procedure can be used to evaluate the cosine or sine of any other angle.

Using the calculator find the values of the following:

(i) cos 67^0 (ii) sin $164^0\ 30'$ (iii) tan 220^0

(iv) sin 1046.82^0 (v) cos $^-236.5^0$

Example 6

Use a calculator to find angles between 0^0 and 360^0 whose:

(a) sine is 0.9064

(b) cosine is –0.4695

(c) tangent is 0.7002

(Give your answers correct to 4 significant figures)

Solution

(a) Step I Key in 0.9064.

Step II Key in the shift function.

Step III Key in sin.

Screen displays 65.012505

$\therefore$ $\sin^{-1} 0.9064 = 65.01^0$ (4 s.f.); angle in the first quadrant.

The other angle is in the second quadrant where sine is positive and this is;

$180^0 - 65.01^0 = 114.99^0$

$= 115.0^0$ (4 s.f.)

(b) Step I Key in 0.4695 followed by $\boxed{+/-}$.

Step II Key in the shift.

Step III Key in cos.

Screen displays 118.00185

$\therefore$ $\cos^{-1} -0.4695 = 118.0^0$ (4 s.f.); an angle in the second quadrant.

The other angle is in the 3rd quadrant, where cosine is also negative, and this is 242^0.

(c) Step I Key in 0.7002.

Step II Key in the shift function.

Step III Key in tan.

Screen displays 34.99971

$\therefore \tan^{-1} 0.7002 = 34.99971 = 35.00^0$ (4 s.f.)

angle in the 1st quadrant.

The other angle is in the third quadrant where tangent is also positive and is 215^0.

Using the calculator, find the values of the following:

(i) $\tan^{-1} 1.192$
(ii) $\cos^{-1} 0.0872$
(iii) $\sin^{-1} -0.8660$

for angles between 0^0 and 360^0

Note:

The calculator used in the workings above is Casio fx-82LB.
Always consult the manual for your calculator.

Exercise 3.4

1. Use the calculator to find the sine, cosine and tangent of each of the following:
   (a) 32.5^0 (b) 64.33^0 (c) 100.42^0 (d) 120.6^0
   (e) 177.2^0 (f) 205.83^0 (g) 313.6^0 (h) 326.42^0
   (i) 550^0 (j) 625.67^0 (k) -175^0 (l) 345.5^0
   (m) -220^0 (n) $-10^0\ 15'$ (p) -80^0
2. (a) Find using the calculator the angles between 0^0 and 360^0 whose sine is:
   (i) 0.8290 (ii) 0.9848 (iii) −0.1736 (iv) −0.9391
   (b) Repeat 2(a) for the range being -180^0 to 180^0 (inclusive).
   (c) Repeat 2(a) for the given values being cosine.
3. Find using a calculator the angles between 0^0 and 360^0 whose tangent is:
   (a) 0.8365 (b) 16.49 (c) −2.1694 (d) −11.075

3.7: Radian Measure

So far, we have been measuring angles in degrees. In this section, we introduce another unit called the **radian**. One radian is the angle subtended at the centre by an arc equal in length to the radius of the circle, see figure 3.13.

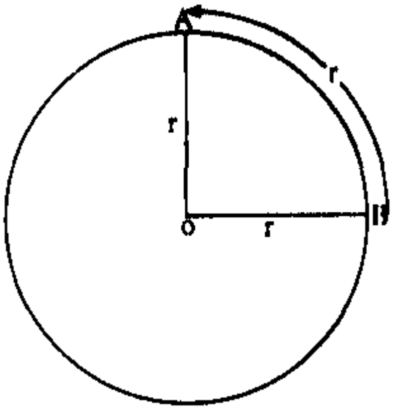


Fig. 3.13

We denote θ radians as θ^C.

The circumference of the circle is $2\pi r$ units. An arc of length r units subtends an angle of 1^C at the centre. Therefore, the angle subtended by the circumference, at the centre is;

$$\left(\frac{2\pi r}{r}\right)^c = 2\pi^C$$

But angle at the centre in degrees is 360^0

Therefore, $2\pi^C = 360^0$

Taking $\pi = 3.142$ (4 sf)

$$\text{It then follows that, } 1^C = \left(\frac{360^0}{2\pi}\right)$$
$$= 57.29 \text{ (4 sf)}$$

Example 7

Convert 125^0 into radians.

Solutions

$57.29^0 = 1^C$

$$\text{Therefore, } 125^0 = \left(\frac{125}{57.29}\right)^c$$
$$= 2.182 \text{ (4 s.f.)}$$

Example 8

Convert the following degrees to radians, giving your answers in terms of π.

(a) 60^0 (b) 405^0

Solution

(a) $360^0 = 2\pi^C$

$$\text{Therefore, } 60^0 = \left(\frac{2\pi}{360} \times 60\right)^c$$

$$= \left(\frac{\pi}{3}\right)^c$$

(b) $$\text{Similarly, } 405^0 = \left(\frac{2\pi}{360} \times 405\right)^c$$

$$= \left(\frac{9\pi}{4}\right)^c$$

Copy and complete the following table

Table 3.5

Degrees	30	45	60	90	180	270	360
Radians			$\frac{\pi}{3}$				2π

Example 9

Convert the following to degrees:

(a) 1.4^C (b) $\left(\frac{2\pi}{5}\right)^C$

Solution

(a) $1^C = 57.29^0$

$$\text{Therefore, } 1.4^C = (57.29 \text{ x } 1.4)^0$$
$$= 80.206$$
$$= 80.21 \text{ (4 s.f.)}$$

(b) We know that $\pi^c = 180^0$

$$\text{Therefore } \left(\frac{2\pi}{5}\right)^C = \left(\frac{2}{5} \times 180\right)^c$$
$$= 72^0$$

Example 10

What is the length of arc which subtends an angle of 0.6 radians at the centre of a circle of radius 20 cm?

Solution

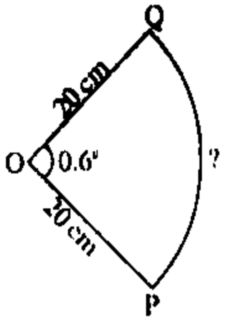


Fig. 3.14

1^C is subtended by 20 cm.

$\therefore$ 0.6^C is subtended by 20×0.6 cm = 12 cm.

Exercise 3.5

1. Convert the following angles in degrees to radians, giving each answer in its simplest form in terms of π:

   (a) 30^0 (b) 135^0 (c) 75^0
   (d) 225^0 (e) 105^0 (f) 720^0
   (g) 900^0 (h) 315^0 (i) 22.5^0

2. Convert the following angles in radians to degrees:

   (a) $\left(\frac{\pi}{5}\right)^c$ (b) $\left(\frac{2}{9}\pi\right)^c$ (c) $\left(\frac{3}{8}\pi\right)^c$
   (d) $\left(\frac{9}{2}\pi\right)^c$ (e) $\left(\frac{11}{3}\pi\right)^c$ (f) $\left(\frac{3}{5}\pi\right)^c$
   (g) $\left(\frac{7}{4}\pi\right)^c$ (h) $\left(\frac{13}{8}\pi\right)^c$ (i) $(3\pi)^c$

3. Convert the following angles in radians to degrees:

   (a) 3^c (b) 2.5^c (c) 1.7^c
   (d) 0.58^c (e) 0.25^c (f) 1.5^c

4. Evaluate:

   (a) $\sin \frac{2}{3}\pi^C$ (b) $\sin \frac{\pi}{12}^C$ (c) $\cos \frac{\pi}{9}^C$
   (d) $\cos 4\pi^C$ (e) $\tan 1.2^C$ (f) $\sin 1.3^C$
   (g) $\cos 0.42^C$ (h) $\tan 6^C$

5. An arc subtends an angle of 0.9 radians. If radius of circle is 13 cm, find the length of the arc.

6. An arc of a circle of radius 4 cm is 5 cm long. Calculate the angle subtended by the arc at the centre:

   (a) in radians.
   (b) in degrees.

3.8: Simple Trigonometric Graphs

Graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0^0 \leq x \leq 360^0$ (or $0^C \leq x \leq 2\pi^C$) can be drawn by choosing suitable values of x and plotting the values of y against the corresponding values of x. The following examples illustrate how to draw the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.

Example 11

Draw the graph of $y = \sin x$ for $0^0 \leq x \leq 360^0$ at an interval of 30^0.

Solution

The values of x and the corresponding values of sin x are given in table 3.6 below.

Table 3.6

x	0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
y = sin x	0	0.5	0.8660	1	0.8660	0.5	0	–0.5	–0.8660	–1	–0.8660	–0.5	0

When we plot and join the ordered pairs (x, sin x), we get as shown in figure 3.15.

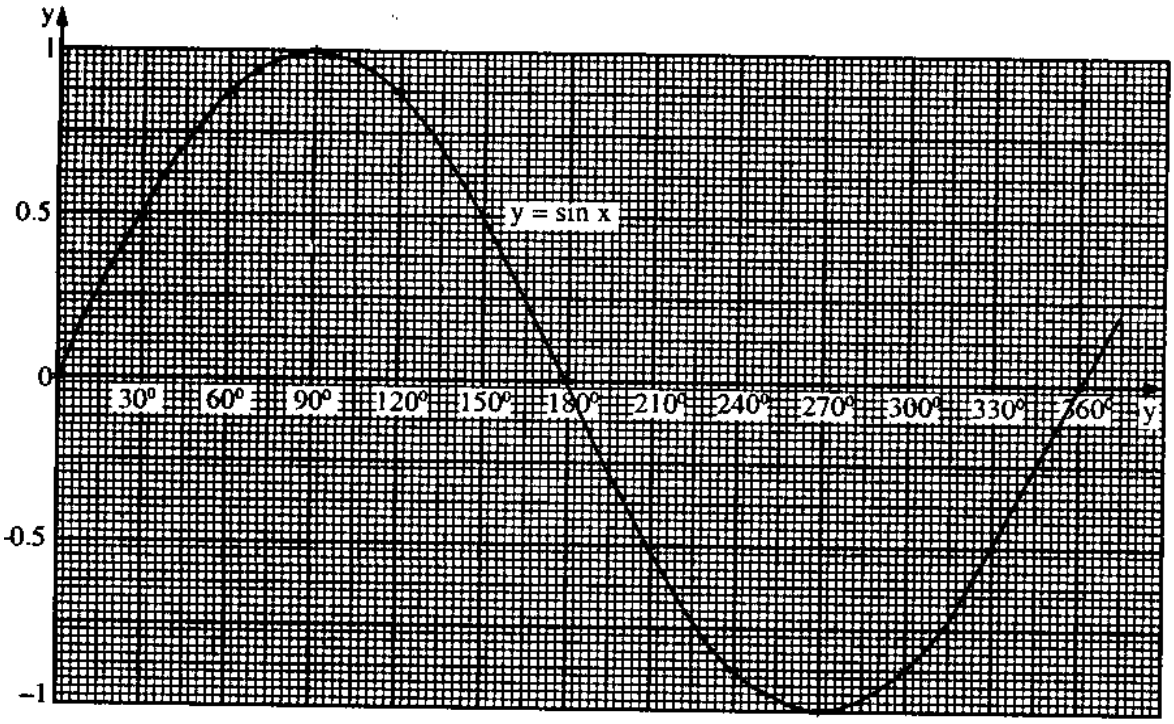


Fig. 3.15

Similarly, draw the graph of $y = \cos x$ for $0^0 \leq x \leq 360^0$ using an interval of 30^0.

Example 12

Draw the graph of $y = \cos x$ for $0^0 \leq x \leq 2\pi^c$ using an internal of $\frac{\pi}{6}^c$.

Solution
The values of x and the corresponding values of cos x are given in table 3.7 below.

Table 3.7

x	0	$\frac{\pi}{6}^c$	$\frac{\pi}{3}^c$	$\frac{\pi}{2}^c$	$\frac{2}{3}\pi^c$	$\frac{5}{6}\pi^c$	π^c
y = cos x	1	0.8660	0.5	0	-0.5	-0.8660	-1

x	$\frac{7}{6}\pi^c$	$\frac{4}{3}\pi^c$	$\frac{3}{2}\pi^c$	$\frac{5}{3}\pi^c$	$\frac{11}{6}\pi^c$	$2\pi^c$
y = cos x	-0.8660	-0.5	0	0.5	0.8660	1

When we plot and join the ordered pairs (x, cos x), we get the graph as shown in figure 3.16.

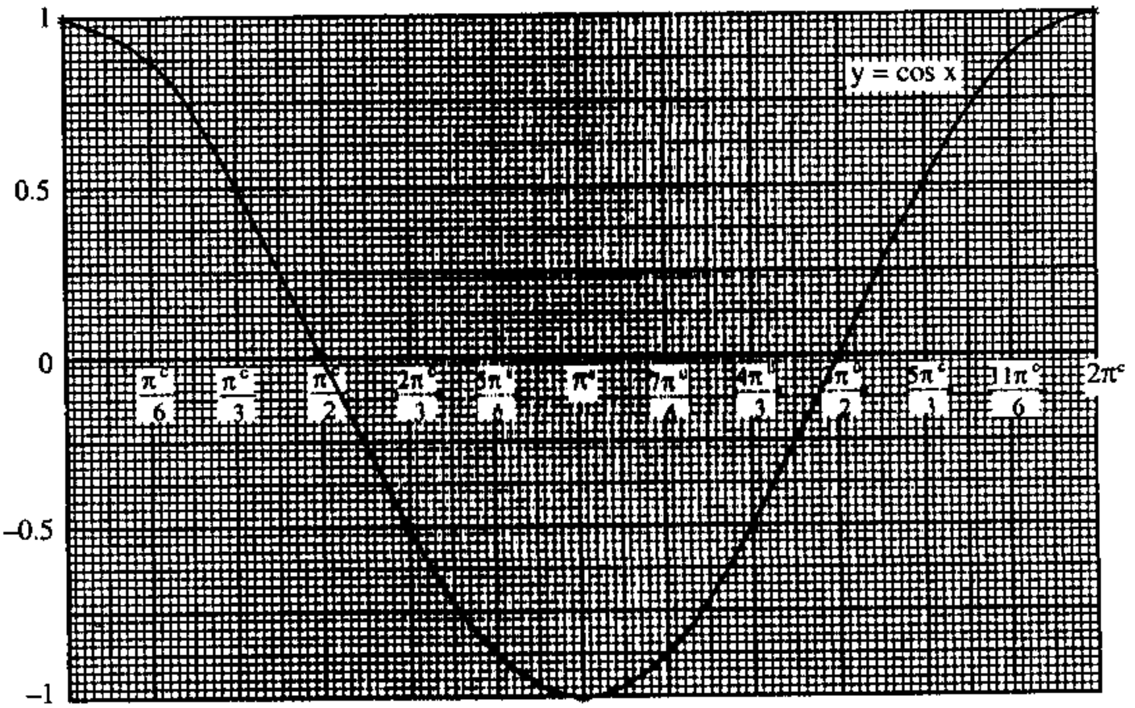


Fig. 3.16

Similarly, draw the graph of y = sin x for $0^0 \le x \le 2\pi^C$ using an interval of $\frac{\pi}{6}^c$.

Example 13

Draw the graph of y = tan x for for $0^0 \le x \le 360^0$ using an interval of 30^0.

Solution

The values of x and the corresponding values of tan x are given in table 3.8 below.

Table 3.8

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0
tan x	0	0.5774	1.732	∞	–1.732	–0.5774	0

x	210^0	240^0	270^0	300^0	330^0	360^0
tan x	0.5774	1.732	∞	–1.732	–0.5774	0

Note that tan 90^0 and tan 270^0 are undefined. The symbol indicating this is ∞. When we plot and join the ordered pairs (x, tan x), we get the graph shown in figure 3.17.

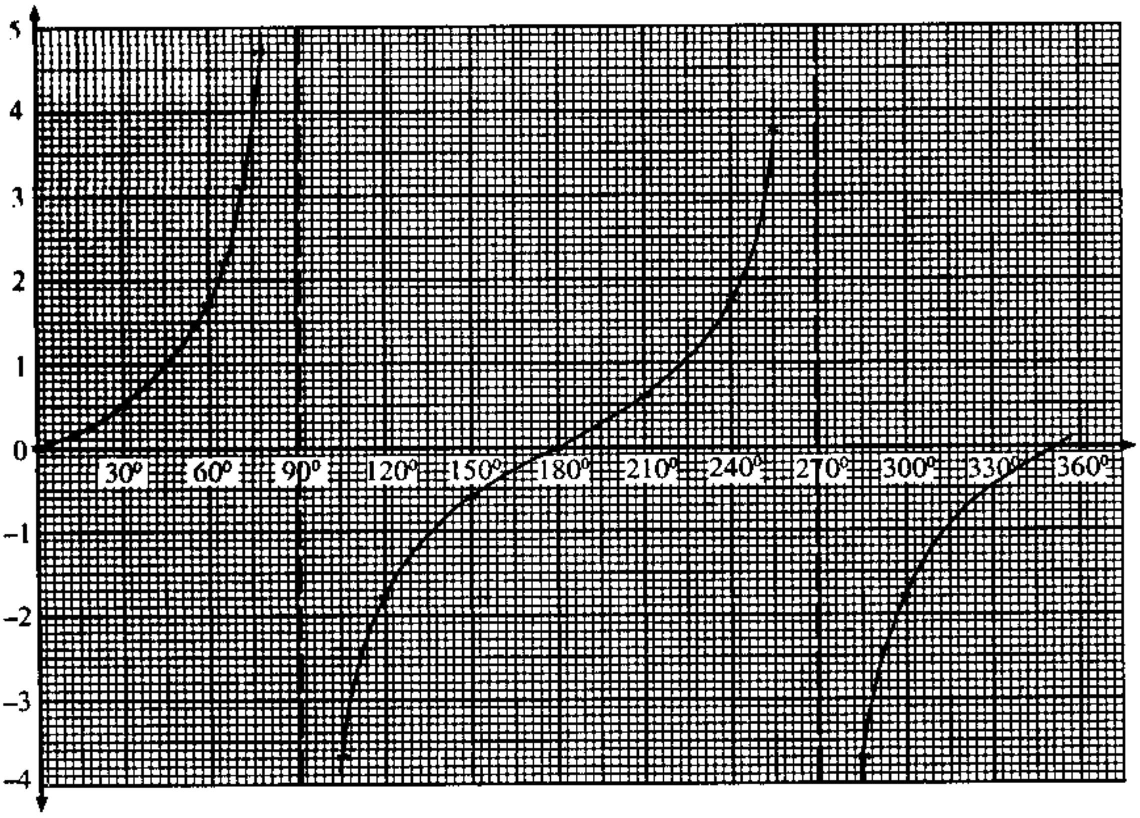


Fig. 3.17

Note that as the value of x approaches **90^0 and 270^0**, tan x becomes very large. Hence, the graph of $y = \tan x$ approaches the lines $x = 90^0$ and $x = 270^0$ without touching them. Such lines are called **asymptotes**. From the graph:

(a) find the values of tan 22.5^0, tan 50^0, tan 135^0, tan 170^0, tan 200^0 and tan 280^0.

(b) find all solutions (in that range) for:

(i) $\tan x = 0.3$

(ii) $\tan x = 1.4$

(iii) $\tan x = -0.9$

Exercise 3.6

1. Draw the graph of y = six x for $0^0 \leq x \leq 360^0$. Use the graph to solve the following equations:
   (a) $\sin x = 0.71$ (b) $\sin x = 0.31$ (c) $\sin x = 0.60$ (d) $\sin x = -0.81$
2. Draw the graph of $y = \cos x$ for $0^c \leq x \leq 2\pi^c$. Use the graph to solve the following equations:
   (a) $\cos x = 0.3$ (b) $\cos x = 0.99$ (c) $\cos x = 0.91$ (d) $\cos x = -0.5$
3. Draw the graph of $y = \tan x$ for $0^0 \leq x \leq 2\pi^c$. Use the graph to solve the following equations:
   (a) $\tan x = 0.7$ (b) $\tan x = -0.2$ (c) $\tan x = 1.2$
4. Draw the graph of $y = \cos x$ for $-180^0 \leq x \leq 180^0$. Use the graph to find the angles whose cosine is:
   (a) 1 (b) 0.4 (c) 0.7 (d) 0.2 (e) 0.6
5. Use the graph of $y = \cos x$ to find the cosine of:
   (a) 45^0 (b) 53^0 (c) 106^0 (d) $\frac{5}{6}\pi^c$ (e) $\frac{7}{9}\pi^C$
6. On the same axes, draw the graphs of $y = \sin x$ and $y = \cos x$ for $0^0 \leq x \leq 360^0$.
   (a) State the co-ordinates of the two points of intersection.
   (b) What transformation maps the graph of $y = \sin x$ onto the graph of $y = \cos x$.
7. Using the range $0^0 \leq x \leq 2\pi^c$, draw the graphs of:
   (a) $y = \sin x + \cos x$ (b) $y = \cos x - \sin x$
8. Using the range $0^0 \leq x \leq 360^0$, draw the graphs of:
   (a) $y = 2 \sin x$ (b) $y = \cos 2x$

3.9: Solution of Triangles

The Sine Rule

The sine rule can be derived by considering the area of a triangle.

In a triangle ABC, the length of the side opposite angle A is denoted by a, the length of the side opposite angle B by b and the length of the side opposite angle C by c, as shown in figure 3.18.

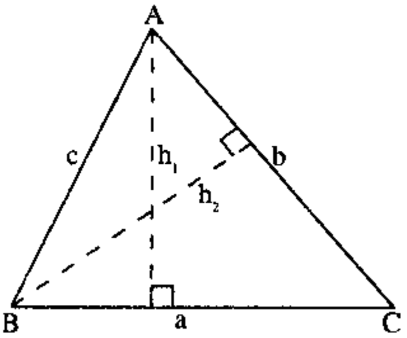


Fig. 3.18

We can find the area of Δ ABC in three different ways:

(i) The area of Δ ABC = $\frac{1}{2}$BC × h_1

But h_1 = c sin B

Therefore, area of Δ ABC = $\frac{1}{2}$a × c sin B

(ii) The area of Δ ABC = $\frac{1}{2}$AC × h_2

But h_2 = a sin C

Therefore, area of Δ ABC = $\frac{1}{2}$ b × a sin C

(iii) Similarly, area of Δ ABC = $\frac{1}{2}$ b × c sin A

Equating the three expressions of the area of Δ ABC we get:

$\frac{1}{2}$bc sin A = $\frac{1}{2}$ac sin B = $\frac{1}{2}$ ab sin C

Dividing each of the expressions by $\frac{1}{2}$abc, we get; $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Taking reciprocals of each of the expressions;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is known as the **sine rule**

If a circle of radius R is circumscribed around the Δ ABC, then;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

The sine rule could be also derived by considering two triangles as in figure 3.19 (a) and (b).

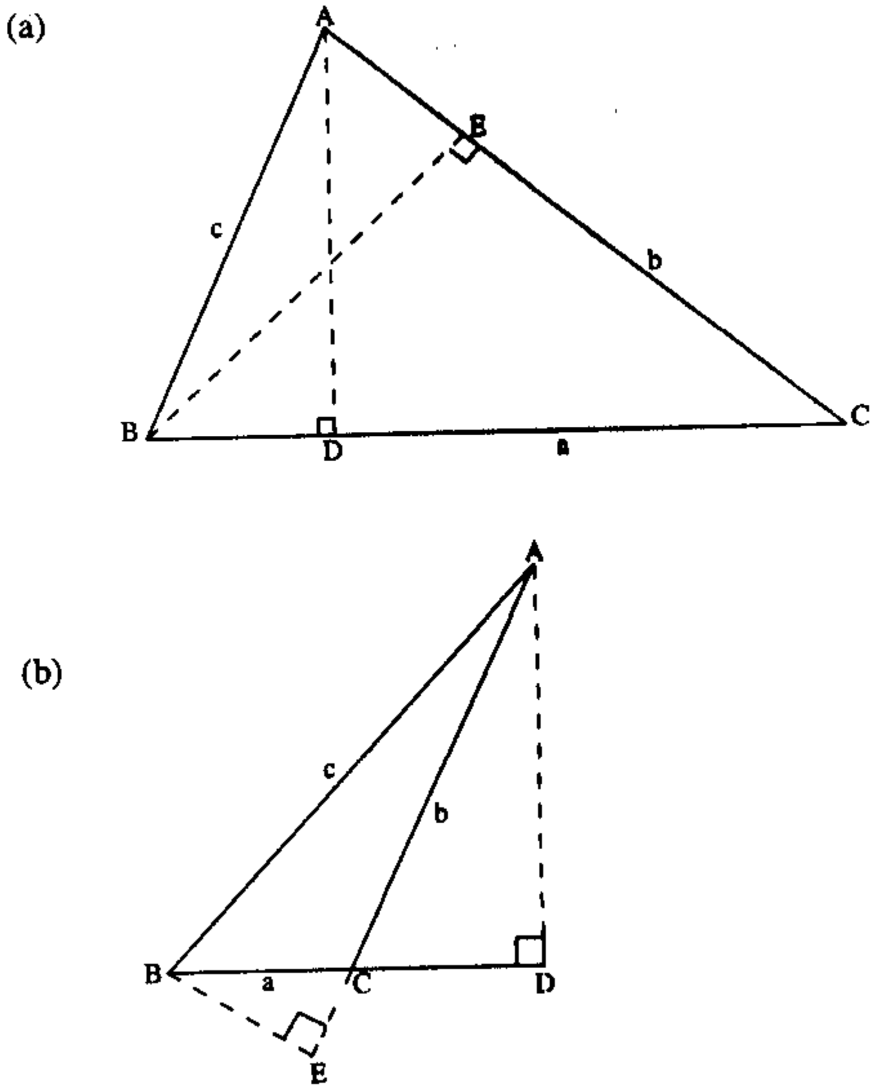


Fig. 3.19

In figure 3.19 (a) and (b), AD and BE are perpendicular to BC and AC respectively.

Sin B = $\frac{AD}{c}$. Therefore, AD = c sin B.

Sin C = $\frac{AD}{b}$. Therefore, AD = b sin C.

Thus, c sin B = b sin C

Hence, $\frac{c}{\sin C} = \frac{b}{\sin B}$

Similarly, BE = c sin A = a sin C

Therefore, $\frac{c}{\sin C} = \frac{a}{\sin A}$

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, which is the sine rule.

The sine rule applies to both acute and obtuse-angled triangles.

Example 14

Solve triangle PQR, given that $R = 42.9^0$, $p = 14.6$ cm and $r = 11.4$ cm.

Solution

To solve a triangle means to find the sides and angles which are not given.
From figure 3.20

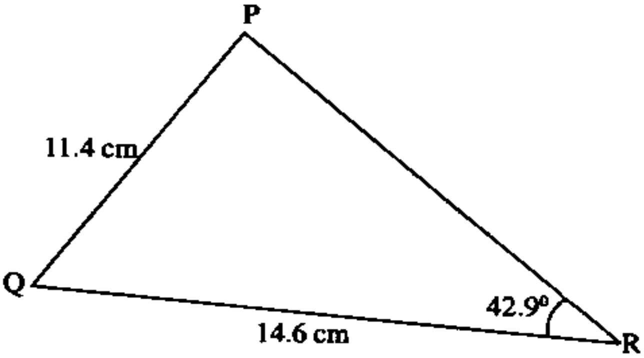


Fig. 3.20

$$\frac{r}{\sin R} = \frac{p}{\sin P}$$

$$\frac{11.4}{\sin 42.9} = \frac{14.6}{\sin P}$$

Hence;

$$\sin P = \frac{14.6 \sin 42.9}{11.4}$$

No	Log
14.6	1.1644
sin 42.9	$\bar{1}.8330$ +
	0.9974
11.4	1.0569 −
0.8720	$\bar{1}.9405$

sin P = 0.8720

Therefore, $P = 60.69^0$

$$\text{Angle } Q = 180^0 - (60.69 + 42.9)$$
$$= 76.41^0$$

$$\frac{q}{\sin 76.41} = \frac{11.4}{\sin 42.9}$$

$$q = \frac{11.4 \sin 76.41}{\sin 42.9}$$

No	Log
11.4	1.0569
sin 76.41	$\bar{1}.9876$ +
	1.0445
sin 42.9^0	$\bar{1}.8330$ –
16.27	1.2115

Therefore, q = 16.27 cm

Example 15

Solve triangle XYZ below, given that XZ = 16.4 cm, $\angle x = 22^0$, $\angle Y = 126^0$

Solution

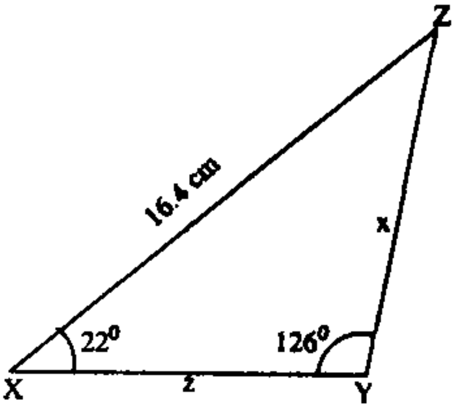


Fig. 3.21

From the figure; $\angle Z = 180^0 - (22^0 + 126^0) = 32^0$

$$\frac{x}{\sin 22^0} = \frac{16.4}{\sin 126^0}$$

$$x = \frac{16.4 \sin 22^0}{\sin 126^0}$$

No	Log
16.4	1.2148
sin 22^0	$\bar{1}.5736$ +
	0.7884
sin 126	$\bar{1}.9080$ −
7.593	0.8804

x = 7.593 cm

$$\frac{z}{\sin 32^0} = \frac{16.4}{\sin 126^0}$$

Hence, $z = \frac{16.4 \sin 32^0}{\sin 126^0}$

$= 10.74$ cm

Verify the above workings using a calculator.

Note:

The sine rule is used when we know:

(i) two sides and a non-included angle of a triangle, or,

(ii) all sides and at least one angle, or,

(iii) all angles and at least one side.

Exercise 3.7

1. Solve the triangle PQR in each of the following cases:
   (a) $Q = 49^0$, $R = 67^0$, $p = 12$ cm
   (b) $Q = 62^0$, $R = 80^0$, $p = 98$ cm
   (c) $P = 24^0$, $q = 38.55$ cm $p = 16$ cm $r = 38.40$ cm
   (d) $Q = 63.2^0$, $r = 5.4$ cm, $q = 9.24$ cm
   (e) $P = 120^0$, $q = 5.6$ cm and $p = 8.5$ cm
2. Solve the triangle ABC in which AB = 5 cm, $\angle ABC = 151^0$ and $\angle BCA = 13^0$.
3. Solve the triangle PQR in which $Q = 35.2^0$, $R = 28.1^0$ and $p = 6.48$ cm.
4. Solve the Δ PQR if $P = 138.5^0$, $R = 22.1^0$ and $p = 77.3$ cm.
5. Solve the Δ PQR if $P = 101.5^0$, $Q = 41.2^0$, $r = 133$ cm.
6. In Δ PQR, calculate the values of P and q if angle $R = 43^0$, $Q = 61^0$ and $r = 5.1$ cm.
7. In Δ XYZ, calculate the possible values of X, Y and y, if $z = 7.6$ cm, $x = 11.11$ cm and $Z = 40^0$.
8. If in Δ PQR, $P = 45^0$, $R = 73^0$ and $p = 3.65$ cm, find Q, q and r.

9. Calculate the shortest side of Δ LMN if angle M = 92°, L = 45° and $l = 6.5$ cm.
10. Two boats P and Q are located 30 km apart, P being due north of Q. An observer at P spots a ship whose bearing he finds as S56° E. From Q, the bearing of the same ship is N38° E. Calculate the distance of the ship from P, and from Q.
11. In triangle PQR, QR = 5 cm and ∠QPR = 60°. Calculate the radius of the circumcircle of the triangle.
12. The distance PQ across a river is to be determined. A point R is 200 m from P and the angles QPR and PRQ are 81° and 75° respectively. Calculate the distance PQ.
13. Calculate PS in figure 3.22 given that QR = 3.63 cm, PR = 2.76 cm, ∠Q = 41.8° and ∠S = 30°.

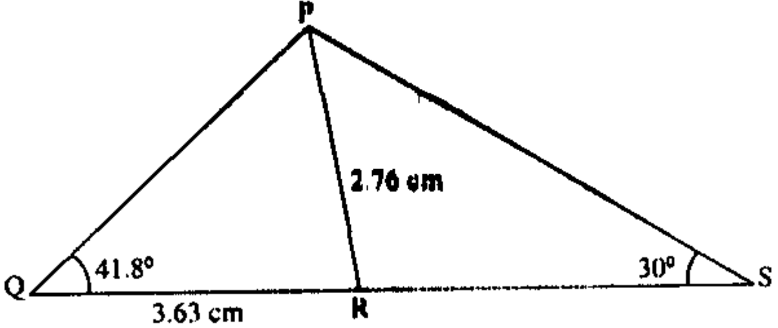


Fig. 3.22

14. A ship starts from a point A on a bearing of 053° and travels for 17 km to a point B. It then changes its course to a bearing of 120° and travels up to a point C. If the bearing of A from C is 290°, find how far C is from A, and the distance from B and C.
15. Calculate the length of MN in figure 3.23 given that KN = 13.6 cm LM = 3.35 cm, ∠KLN = 81°, ∠KNL = 66° and ∠LMN = 130°.

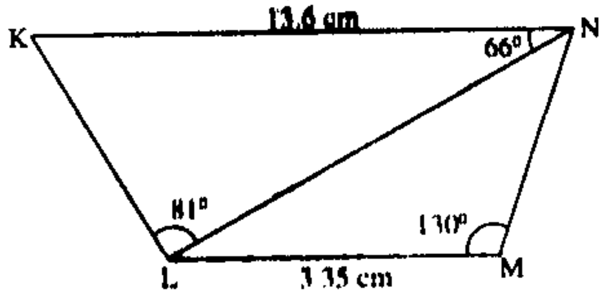


Fig. 3.23

The Cosine Rule

Figure 3.24 (a) and (b) show two triangle, one acute-angled and the other obtuse-angled at C.

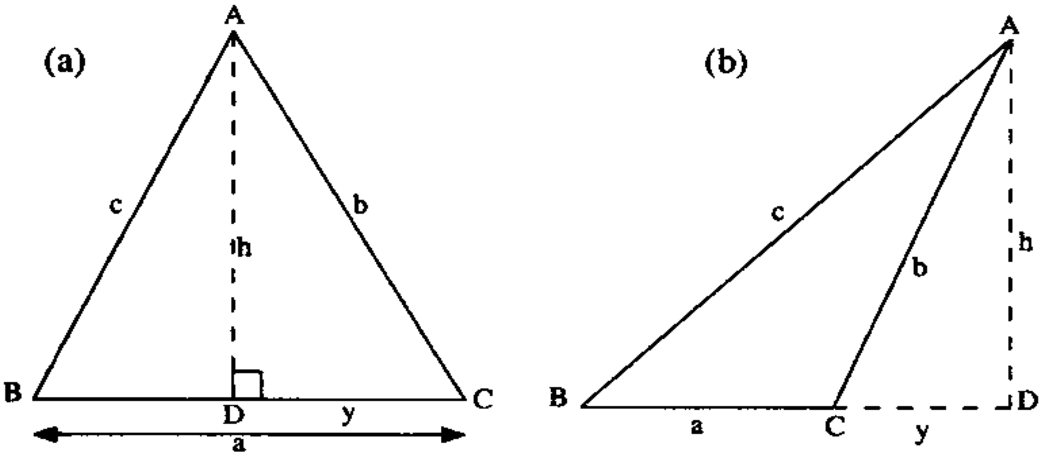


Fig. 3.24

AD is perpendicular to BC produced in the case of figure 3.24 (b).

In figure 3.24 (a);

$c^2 = (a - y)^2 + h^2$ (Pythagoras' theorem)

$= a^2 - 2ay + y^2 + h^2$

But $y^2 + h^2 = b^2$ (Pythagoras' theorem)

$\therefore c^2 = a^2 - 2ay + b^2$

In Δ ACD;

$\cos C = \frac{y}{b}$

Therefore, $y = b \cos C$

Substituting for y in $c^2 = a^2 + b^2 - 2ay$;

$c^2 = a^2 + b^2 - 2ab \cos C$

This is known as the **cosine rule.**

In figure 3.24 (b), $c^2 = (a + y)^2 + h^2$ (Pythagoras' theorem)

$= a^2 + 2ay + y^2 + h^2$

But $y^2 + h^2 = b^2$

Therefore, $c^2 = a^2 + 2ay + b^2$

In Δ ACD;

$\text{Cos } C = \frac{y}{b}$. But $\angle$ACD is supplementary to $\angle$ ACB.

Therefore, in Δ ACB, $\cos C = -\frac{y}{b}$. Hence, $y = -b \cos C$

Substituting for y in $c^2 = a^2 + b^2 + 2ay$;

$c^2 = a^2 + b^2 + 2a(-b\cos C)$

$= a^2 + b^2 - 2ab\cos C$

This is the **cosine rule.**

Similarly, $a^2 = b^2 + c^2 - 2bc \cos A$, and $b^2 = a^2 + c^2 - 2ac \cos B$

Example 16

Find PQ in figure 3.25.

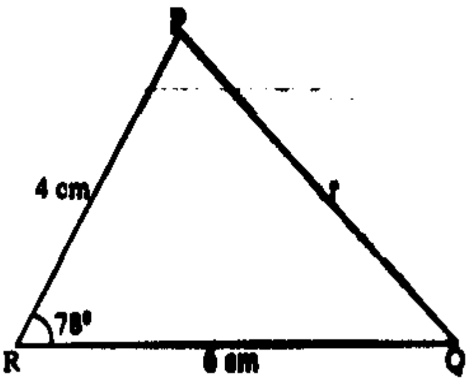


Fig. 3.25

Solution

Using the cosine rule;

$r^2 = q^2 + p^2 - 2qp \cos R$

$= 4^2 + 6^2 - 2 \times 4 \times 6 \cos 78^0$

$= 16 + 36 - 48 \cos 78^0$

$= 52 - 48 \times 0.2079$

$= 52 - 9.979$

$= 42.02$ cm

Therefore, r = 6.482 cm.

Example 17

Find ∠ACB in figure 3.26.

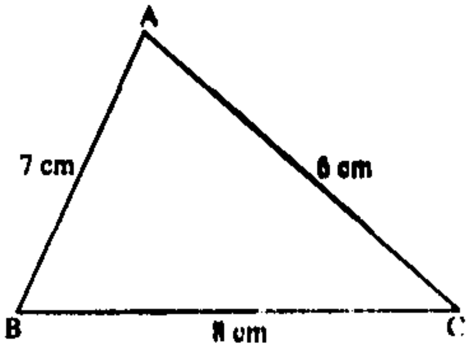


Fig. 3.26

Solution

Using the cosine rule;

$c^2 = a^2 + b^2 - 2ab \cos C$

$7^2 = 8^2 + 6^2 - 2 \text{ x } 8 \text{ x } 6 \cos C$

$49 = 64 + 36 - 96 \cos C$

$\cos C = \frac{51}{96}$

$= 0.5313$

Therefore, $C = 57.91^0$

Note:

The cosine rule is used when we know:

(i) two sides and an included angle, or,

(ii) all three sides of a triangle.

Example 18

In figure 3.27, PQ = 12 cm, QR = 10 cm and $\angle PQR = 143.7^0$. Find PR.

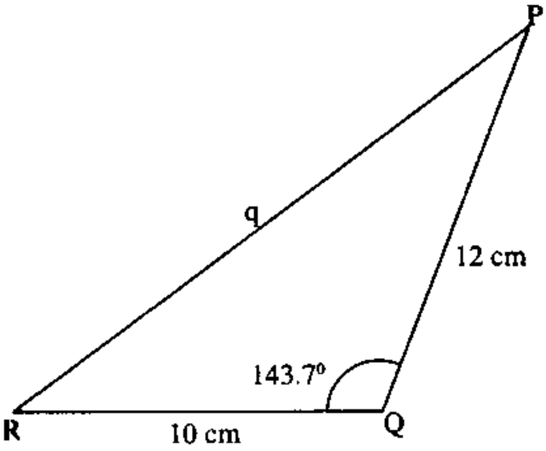


Fig. 3.27

Solution

Using the cosine rule;

$$q^2 = p^2 + r^2 - 2pr \cos Q$$
$$= 10^2 + 12^2 - 2 \times 10 \times 12 \cos 143.7^0$$
$$= 100 + 144 - 240 \{-\cos (180^0 - 143.7^0)\}$$
$$= 244 + 240 \cos 36.3^0$$
$$= 244 + 240 \times 0.8059$$
$$= 244 + 193.42$$
$$= 437.42$$

Therefore, q = 20.91 cm.

Example 19

Figure 3.28 shows a triangle XYZ in which x = 13.4 cm, z = 5 cm and $\angle XYZ = 57.7^0$. Solve the triangle.

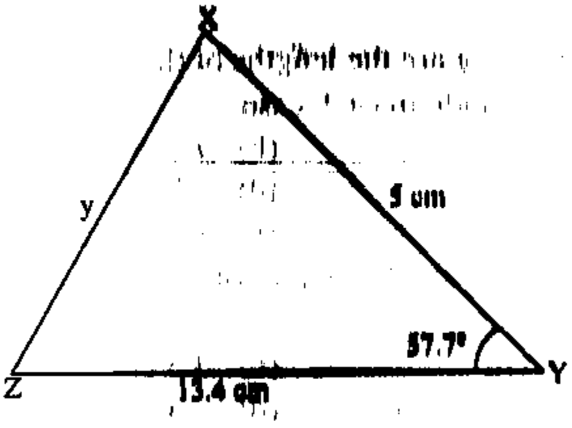


Fig. 3.28

Solution

In this case, we should find the value of y, ∠XZY and ∠YXZ.

$y^2 = (13.4)^2 + 5^2 - 2 \times 13.4 \times 5 \cos 57.7^0$

$= 179.6 + 25 - 134 \times 0.5344$

$= 204.6 - 71.61$

$= 132.99$

$\therefore$ y = 11.53 cm.

Using sine rule;

$$\frac{5}{\sin Z} = \frac{11.53}{\sin 57.7}$$

$5 \sin 57.7 = 11.53 \sin Z$

$$\sin Z = \frac{5 \sin 57.7}{11.53}$$

No	Log
5	0.6990
sin 57.7	$\bar{1}$.9270 +
	0.6260
11.53	1.0618 −
0.3666	$\bar{1}$.5642

$\sin Z = 0.3666$

Therefore, $Z = 21.51^0$.

Finally, $\angle YXZ = 180^0 - (21.51^0 + 57.7)$

$= 180^0 - 79.21^0$

$= 100.79^0$

$= 100.8^0$ (4 s.f.)

Exercise 3.8

1. Given that the following are the lengths of the sides of a triangle, Calculate the largest angle in each case:

(a)	4 cm, 6 cm, 9.2 cm	(b)	7.2 cm, 10.1 cm, 5.5 cm
(c)	4 cm, 2 cm, 3 cm	(d)	4.7 cm, 6.2 cm, 8.3 cm
(e)	7.8 cm, 5 cm, 11 cm	(f)	6.5 cm, 8.4 cm. 6.5 cm

2. Given that the following are the lengths of the sides of a triangle, calculate the smallest angle in each case:

(a)	3 cm, 4 cm, 5 cm	(b)	3.5 cm, 6.5 cm, 5 cm
(c)	7.5 cm, 8.1 cm, 11.3 cm	(d)	3.45 cm, 7.4 cm, 9.3 cm
(e)	8.82 cm, 5.92 cm, 5.02 cm.		

3. In figure 3.29, C is the centre of the circle. $\angle PQR = 112^0$, PQ = 5.41 cm and QR = 4.73 cm. Find the length PR and the area of the circle.

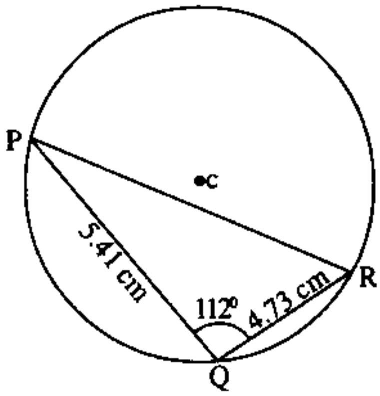


Fig. 3.29

4. In a triangle LMN, L = 81^0, n = 4.3 cm and m = 3.5 cm. Calculate the length *l* and the angles M and N.
5. The perimeter of a triangular field is 120 m. Two of the sides are 21 m and 40 m. Calculate the largest angle of the field.
6. Solve Δ PQR in which p = 10.4 cm, q = 25.6 cm and R = 116^0.
7. Determine the length of the side opposite the given angle in each case of the following triangles:
   (a) Δ ABC in which a = 13 cm, b = 15 cm and C = 71^0.
   (b) Δ PQR in which p = 8.1 cm, q = 6.4 cm and R = 37^0.
   (c) Δ XYZ in which x = 3.3 cm, z = 10.0 cm and Y = 129^0.
   (d) Δ PQR in which r = 1.89 cm, q = 4.12 cm and P = 79^0.
8. Solve the triangle LMN in which m = 3.42 cm, n = 7.43 and L = 140^0.
9. In figure 3.30 PQRS is a trapezium. Calculate the angles of triangles PQR and PRS

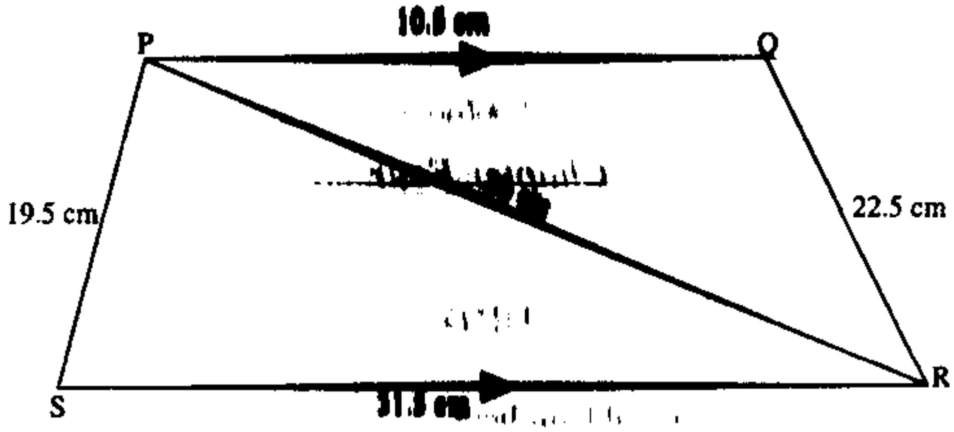


Fig. 3.30

10. A ship sails due north from a point A for 62 km to a point B. It changes its course to N 47°E and sails up to a point C. Find the distance from C to A if C is N 25° E of A.
11. Three ships X, Y and Z are approaching a harbour H. X is 16 km from the harbour on a bearing of 090°. Y is 14 km from the harbour on a bearing of 130°, and Z is 26.31 km to the west of Y and on a bearing of 240° from the harbour. Calculate:
    (a) the distance between X and Y.
    (b) the distance of Z from the harbour.
    (c) the distance between X and Z.
12. The angle elevation of the top of a flag post from a point A on level ground is 13°. The angle of elevation of the top of the flag post from another point B nearer the flag post and 120 m from A is 30°. B is between A and the bottom of the flag post and the three points are collinear. Find
    (a) the distance from the point B to the top of the flag post.
    (b) the height of the flag post.

Chapter Four

SURDS

4.1: Rational and Irrational Numbers

A rational number is a number which can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. The integers p and q must not have common factors other than 1. Numbers such as 2, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$, $\sqrt{4}$, $\sqrt[3]{8}$ and $\sqrt{9}$ are examples of rational numbers. Terminating and recurring decimals are also rational numbers since they can be expressed in the form $\frac{p}{q}$.

Irrational Numbers

Numbers such as π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $3\sqrt{10}$ cannot be written in the form $\frac{p}{q}$.

Such numbers are called **irrational numbers.**

Classify the following numbers as rational or irrational:

(i) $\sqrt{11}$ (ii) $\sqrt{8}$ (iii) $\sqrt{16}$ (iv) $\sqrt{25}$

(v) $\sqrt[4]{32}$ (vi) $\sqrt[4]{16}$ (vii) $\sqrt{21}$ (viii) $\sqrt[3]{27}$

Surds

Consider expressions such as $\sqrt{9}$, $\sqrt{\frac{16}{9}}$, $16^{\frac{1}{2}}$, $\sqrt{0.09}$ and $\sqrt[3]{8}$. These numbers **have exact square** roots or cube roots.

Thus, $\sqrt{9} = \pm 3$, $16^{\frac{1}{2}} = \pm 4$, $\sqrt{0.09} = \pm 0.3$ and $\sqrt[3]{8} = 2$.

However, expressions such as $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{36}$, $\sqrt[3]{16}$ do not have exact square roots or cube roots. Such numbers are called **surds.** All surds are irrational numbers. Although a surd is an irrational number, the number under the root sign is a rational number.

The product of a surd and a rational number is called a mixed surd. Examples are;

$2\sqrt{3}$, $4\sqrt{7}$ and $\frac{1}{3}\sqrt{2}$

4.2: Order of Surds

$\sqrt{3}$, $\sqrt{6}$, $\sqrt{10}$, $\sqrt{12}$ are surds of order 2.

$\sqrt[3]{2}$, $\sqrt[3]{6}$, $\sqrt[3]{10}$, $\sqrt[3]{30}$ are surds of order 3.

$\sqrt[4]{2}$, $\sqrt[4]{64}$, $\sqrt[4]{100}$ are surds of order 4.

In general, if $\sqrt[n]{a}$ is a surd, then the order of the surd is n.

4.3: Simplification of Surds

A surd can be reduced to its lowest term possible, as follows;

Example 1

Simplify: (a) $\sqrt{18}$ (b) $\sqrt{48}$ (c) $\sqrt{72}$

Solution

(a) $\sqrt{18} = \sqrt{9 \times 2}$

$= \sqrt{9} \times \sqrt{2}$

$= 3\sqrt{2}$

(b) $\sqrt{48} = \sqrt{16 \times 3}$

$= \sqrt{16} \times \sqrt{3}$

$= 4\sqrt{3}$

(c) $\sqrt{72} = \sqrt{36 \times 2}$

$= \sqrt{36} \times \sqrt{2}$

$= 6\sqrt{2}$

Simplify each of the following to the lowest term:

(i) $\sqrt{50}$ (ii) $\sqrt{200}$ (iii) $\sqrt{12}$ (iv) $\sqrt{125}$ (v) $\sqrt{54}$

4.4: Operations on Surds

The four basic operation of addition, subtraction, division and multiplication apply to surds in the same way as in algebra.

Addition and Subtraction

Surds can only be added or subtracted if they are of the same order and the number under the root sign is the same. Otherwise, the surd is in its simplest form.

Example 2

Evaluate:

(a) $\sqrt{2} + \sqrt{2}$ (b) $3\sqrt{3} + 5\sqrt{3}$ (c) $5\sqrt{3} - 2\sqrt{3}$ (d) $3\sqrt{6} + 4\sqrt{10}$

Solution

(a) $\sqrt{2} + \sqrt{2}$

Let a = $\sqrt{2}$

$\therefore \sqrt{2} + \sqrt{2} = a + a$

$= 2a$

But a = $\sqrt{2}$

$\therefore \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

(b) $3\sqrt{5} + 5\sqrt{3}$

Let a = $\sqrt{3}$

$\therefore 3\sqrt{3} + 5\sqrt{3} = 3a + 5a$

$= 8a$

But a = $\sqrt{3}$

$\therefore 3\sqrt{3} + 5\sqrt{3} = 8\sqrt{3}$

(c) $5\sqrt{3} - 2\sqrt{3}$

Let a = $\sqrt{3}$

$\therefore 5\sqrt{3} - 2\sqrt{3} = 5a - 2a$

$= 3a$

But a = $\sqrt{3}$

$5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$

(d) $3\sqrt{6} + 4\sqrt{10} = 3\sqrt{6} + 4\sqrt{10}$; Expression already in its simplest form.

In general, (i) $a\sqrt{n} + b\sqrt{n} = (a + b)\sqrt{n}$

(ii) $a\sqrt{n} - b\sqrt{n} = (a - b)\sqrt{n}$

Multiplication and Division

Surds of the same order can be multiplied or divided irrespective of the number under the root sign.

Example 3

Evaluate:

(a) $\sqrt{2} \times \sqrt{2}$ (b) $\sqrt{2} \times \sqrt{3}$ (c) $2\sqrt{6} \times 7\sqrt{10}$

(d) $\sqrt{10} \div \sqrt{5}$ (e) $2\sqrt{15} \div \sqrt{3}$

Solution

(a) $\sqrt{2} \times \sqrt{2} = \sqrt{2 \times 2}$

$= \sqrt{4}$

$= 2$

(b) $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$

$= \sqrt{6}$

(c) $2\sqrt{6} \times 7\sqrt{10} = (2 \times 7)\sqrt{6 \times 10}$

$= 14\sqrt{60}$

$= 14\left(\sqrt{4} \times \sqrt{15}\right)$

$= 14 \times 2\sqrt{15}$

$= 28\sqrt{15}$

(d) $\sqrt{10} \div \sqrt{5} = \sqrt{\frac{10}{5}}$

$= \sqrt{2}$

(e) $2\sqrt{15} \div \sqrt{3} = 2\sqrt{\frac{15}{3}}$

$= 2\sqrt{5}$

In general; (i) $\sqrt{m} \times \sqrt{n} = \sqrt{mn}$

(ii) $\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$

Note:

$\sqrt{(a \pm b)} \neq \sqrt{a} \pm \sqrt{b}$

Example 4

Simplify:

(a) $\left(\sqrt{2} + \sqrt{3}\right)^2$ (b) $\left(2 - \sqrt{5}\right)^2$

(c) $\left(2 - \sqrt{3}\right)\left(2 + \sqrt{3}\right)$ (d) $\left(2\sqrt{3} + \sqrt{7}\right)\left(\sqrt{3} + 4\sqrt{11}\right)$

Solution

(a)
$$\begin{aligned}\left(\sqrt{2}+\sqrt{3}\right)^2 &= \left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)\\ &= \sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{3}\left(\sqrt{2}+\sqrt{3}\right)\\ &= \sqrt{4}+\sqrt{6}+\sqrt{4}+\sqrt{6}\\ &= 2+2\sqrt{6}+3\\ &= 5+2\sqrt{6}\end{aligned}$$

(b)
$$\begin{aligned}\left(2-\sqrt{5}\right)^2 &= \left(2-\sqrt{5}\right)\left(2-\sqrt{5}\right)\\ &= 2\left(2-\sqrt{5}\right)-\sqrt{5}\left(2-\sqrt{5}\right)\\ &= 4-2\sqrt{5}-2\sqrt{5}+\sqrt{25}\\ &= 4-4\sqrt{5}+5\\ &= 9-4\sqrt{5}\end{aligned}$$

(c)
$$\begin{aligned}\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right) &= 2\left(2+\sqrt{3}\right)-\sqrt{3}\left(2+\sqrt{3}\right)\\ &= 4+2\sqrt{3}-2\sqrt{3}-\sqrt{9}\\ &= 4-3\\ &= 1\end{aligned}$$

(d)
$$\begin{aligned}\left(2\sqrt{3}+\sqrt{7}\right)\left(\sqrt{3}+4\sqrt{11}\right) &= 2\sqrt{3}\left(\sqrt{3}+4\sqrt{11}\right)+\sqrt{7}\left(\sqrt{3}+4\sqrt{11}\right)\\ &= 2\sqrt{9}+8\sqrt{33}+\sqrt{21}+4\sqrt{77}\\ &= 2\times 3+8\sqrt{33}+\sqrt{21}+4\sqrt{77}\\ &= 6+8\sqrt{33}+\sqrt{21}+4\sqrt{77}\end{aligned}$$

Exercise 4.1

1. Express each of the following surds in its simplest form:

(a) $\sqrt{8}$ (b) $\sqrt{12}$ (c) $\sqrt{32}$ (d) $\sqrt{52}$

(e) $\sqrt{1\,000}$ (f) $\sqrt{250}$ (g) $\sqrt{128}$ (h) $\sqrt{54}$

2. Copy and complete the table:

$4\sqrt{3}$	$\sqrt{4^2(\sqrt{3})^2}$	$\sqrt{16 \times 3}$	$\sqrt{48}$
$2\sqrt{5}$	$\sqrt{2^2(\sqrt{5})^2}$	$\sqrt{4 \times 5}$	$\sqrt{20}$
$6\sqrt{3}$	—	—	$\sqrt{108}$
—	$\sqrt{7^2(\sqrt{2})^2}$	—	$\sqrt{98}$
$13\sqrt{10}$	—	$\sqrt{169 \times 10}$	—
$20\sqrt{10}$	—	—	—

3. Simplify:

(a) $2\sqrt{3} + 5\sqrt{3}$ (b) $3\sqrt{7} + 10\sqrt{7}$ (c) $2\sqrt{7} + \frac{1}{2}\sqrt{7}$

(d) $5\sqrt{11} + \sqrt{11}$ (e) $3\sqrt{23} + 5\sqrt{23} - 14\sqrt{23}$

(f) $2\sqrt{x} + 3\sqrt{x}$ (g) $2\sqrt{9x} + 5\sqrt{3x}$

(h) $4\sqrt{18} + 5\sqrt{72} - \sqrt{108}$ (i) $3\sqrt{x^2} + 7\sqrt{(4x^2)} - 21\sqrt{(16x^2)}$

(j) $3\sqrt{19} - 2\sqrt{19}$ (k) $6\sqrt{12} - 3\sqrt{10}$ (l) $7\sqrt{24} - 9\sqrt{24}$

(m) $16\sqrt{40}$ (n) $8\sqrt{32} - 2\sqrt{32}$ (p) $\frac{1}{2}\sqrt{6} - \frac{1}{3}\sqrt{6}$

(q) $\frac{4}{3}\sqrt{125} - 3\sqrt{5}$ (r) $10\sqrt{216} - \frac{3}{4}\sqrt{6}$ (s) $\frac{4}{3}\sqrt{27} - \frac{1}{2}\sqrt{3}$

(t) $2\sqrt{343} - 10\sqrt{7}$

4. Simplify:

(a) $\sqrt{5} \times \sqrt{2}$ (b) $2\sqrt{3} \times 5\sqrt{7}$ (c) $\sqrt{11} \times \sqrt{9}$

(d) $3\sqrt{13} \times 5\sqrt{2}$ (e) $6\sqrt{4} \times \frac{1}{2}\sqrt{3}$ (f) $8\sqrt{x} \times 2\sqrt{y}$

(g) $2\sqrt{3} \times \frac{1}{5}\sqrt{5} \times 8\sqrt{23}$ (h) $(\sqrt{5})^2$ (i) $\sqrt{24} \times \sqrt{6}$

(j) $2\sqrt{8} \times 5\sqrt{3}$ (k) $\sqrt{24} \times \sqrt{6}$ (l) $\sqrt{72} \times \sqrt{2}$

(m) $\sqrt{81} \times \sqrt{7}$ (n) $\sqrt{225} \times \sqrt{15}$ (p) $\sqrt{169} \times \sqrt{14}$

(q) $\sqrt{196} \times \sqrt{5}$ (r) $\sqrt{625} \times \sqrt{7}$ (s) $2\sqrt{28} \times 3\sqrt{42}$

5. **Simplify:**

(a) $\sqrt{(56 \times 4)}$ (b) $\sqrt{72} \times \frac{1}{2}$ (c) $5\sqrt{(125 \times 6)}$

(d) $\sqrt{400}$ (e) $\sqrt{124}$ (f) $\sqrt{96}$

(g) $\sqrt{(92 \times 16)}$ (h) $\sqrt{9} \times \sqrt{24}$ (i) $\sqrt{(64 \times 13)}$

(j) $16x^2\sqrt{9x^2}$ (k) $\sqrt{(6 \times 6)}$ (l) $\sqrt{(49 + 4)}$

(m) $\sqrt{(625 \times 9)}$ (n) $\sqrt{(169 \times 18)}$ (p) $\sqrt{(156 \times 25)}$

(q) $\sqrt{(72 \times 196)}$ (r) $\sqrt{(324 \times 3)}$ (s) $\sqrt{(150 \times 360)}$

(t) $\sqrt{(125 \times 2)}$ (u) $\sqrt{(81 - 16)}$

6. **Simplify:**

(a) $\sqrt{\frac{81}{36}}$ (b) $\sqrt{\frac{625}{100}}$ (c) $\sqrt{\frac{196}{81}}$ (d) $\sqrt{\frac{49}{16}}$

(e) $\sqrt{\frac{243}{26}}$ (f) $\sqrt{\frac{64}{4}}$ (g) $\sqrt{\frac{128}{196}}$ (h) $\sqrt{\frac{360}{160}}$

(i) $\sqrt{\frac{48}{16}}$ (j) $\frac{3\sqrt{24}}{4\sqrt{25}}$ (k) $\frac{\sqrt{72}}{\sqrt{32}}$ (l) $\frac{-2\sqrt{200}}{3\sqrt{27}}$

(m) $\frac{17\sqrt{200}}{6\sqrt{400}}$ (n) $\frac{\sqrt{648}}{\sqrt{72}}$ (p) $\frac{16x^2y^6\sqrt{18x^8y^4}}{8x^3y^2\sqrt{6xy}}$ (q) $\sqrt{\frac{72 \times 18}{41 \times 16}}$

(r) $\frac{\sqrt{\frac{49}{12}}}{\sqrt{\frac{28}{4}}}$ (s) $\frac{\sqrt{\frac{256}{72}}}{\sqrt{\frac{244}{16}}}$

7. **Evaluate, leaving your answer in its simplest form:**

(a) $\left(2 + \sqrt{3}\right)\left(2 + \sqrt{3}\right)$ (b) $(\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3})$

(c) $(\sqrt{5} - \sqrt{3})(\sqrt{3} + \sqrt{2})$ (d) $(\sqrt{5} - \sqrt{2})(\sqrt{5} - \sqrt{2})$

(e) $\left(\sqrt{6} - \sqrt{8}\right)\left(\sqrt{7} - \sqrt{6}\right)$ (f) $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{6})$

(g) $(\sqrt{3} - \sqrt{5})(\sqrt{10} + \sqrt{2})$ (h) $\left(\sqrt{20} - \sqrt{15}\right)\left(\sqrt{2} + \sqrt{5}\right)$

(i) $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{10}+\sqrt{2}\right)$ (j) $\left(2+\sqrt{3}\right)\left(2+\sqrt{5}\right)$

(k) $\left(\sqrt{7}+4\right)\left(\sqrt{7}-4\right)$ (l) $\left(4\sqrt{3}-2\sqrt{3}\right)\left(4\sqrt{3}+6\sqrt{3}\right)$

(m) $\left(\sqrt{125}+\sqrt{5}\right)\left(\sqrt{3}-\sqrt{7}\right)$ (n) $\left(\sqrt{149}+\sqrt{160}\right)\left(\sqrt{3}-\sqrt{6}\right)$

4.5: Rationalising the Denominator

The process of expressing a fraction having a surd in the denominator in such a way that there is no such a surd is called rationalising the denominator.

Example

Rationalise: (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{7}}$

Solution

(a)
$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5\times5}} = \frac{\sqrt{5}}{\sqrt{25}} = \frac{\sqrt{5}}{5}$$

(b)
$$\frac{1}{\sqrt{7}} = \frac{1}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{\sqrt{7\times7}} = \frac{\sqrt{7}}{\sqrt{49}} = \frac{\sqrt{7}}{7}$$

Consider the expressions;

(a) $(5-2\sqrt{7})(5+2\sqrt{7})$ (b) $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

Expanding and simplifying;

(a)
$$\begin{aligned}(5-2\sqrt{7})(5+2\sqrt{7}) &= 5(5+2\sqrt{7}) - 2\sqrt{7}(5+2\sqrt{7})\\ &= 25+10\sqrt{7}-10\sqrt{7}-4\sqrt{49}\\ &= 25-4\times7\\ &= 25-28\\ &= -3\end{aligned}$$

(b)
$$\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right) = \sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)-\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$$

$$= \left(\sqrt{9}+\sqrt{6}\right)-\left(\sqrt{6}-\sqrt{4}\right)$$
$$= 3-2$$
$$= 1$$

Note that each expression reduces to a rational number. If the product of two surds gives a rational number, then the surds are called **conjugates** of each other or simply **conjugate surds**.

$(5-2\sqrt{7})$ is the conjugate of $(5+2\sqrt{7})$ and vice versa. Similarly, $\left(\sqrt{3}-\sqrt{2}\right)$ is the conjugate of $\left(\sqrt{3}+\sqrt{2}\right)$ and vice versa.

In general, the conjugate of $\left(\sqrt{a}-\sqrt{b}\right)$ is $\left(\sqrt{a}+\sqrt{b}\right)$.

Example 5

Simplify by rationalising the denominator:

(a) $\dfrac{8}{\sqrt{5}+\sqrt{2}}$ (b) $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{5}-\sqrt{2}}$

Solution

(a)
$$\frac{8}{\sqrt{5}+\sqrt{2}} = \frac{8}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$
$$= \frac{8(\sqrt{5}-\sqrt{2})}{\sqrt{5}(\sqrt{5}-\sqrt{2})+\sqrt{2}(\sqrt{5}-\sqrt{2})}$$
$$= \frac{8(\sqrt{5}-\sqrt{2})}{\sqrt{25}-\sqrt{10}+\sqrt{10}-\sqrt{4}}$$
$$= \frac{8(\sqrt{5}-\sqrt{2})}{5-2}$$
$$= \frac{8(\sqrt{5}-\sqrt{2})}{3}$$

(a)
$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{6}-\sqrt{3}} = \frac{(\sqrt{2}+\sqrt{3})}{(\sqrt{6}-\sqrt{3})} \times \frac{(\sqrt{6}+\sqrt{3})}{(\sqrt{6}+\sqrt{3})}$$
$$= \frac{\sqrt{2}(\sqrt{6}+\sqrt{3})+\sqrt{3}(\sqrt{6}+\sqrt{3}}{\sqrt{6}(\sqrt{6}+\sqrt{3})-\sqrt{3}(\sqrt{6}+\sqrt{3})}$$
$$= \frac{\sqrt{12}+\sqrt{6}+\sqrt{18}+\sqrt{9}}{\sqrt{36}+\sqrt{18}-\sqrt{18}-\sqrt{9}}$$

$$= \frac{\sqrt{4}\times\sqrt{3}+\sqrt{6}+\sqrt{9}\times\sqrt{2}+\sqrt{9}}{6-3}$$

$$= \frac{2\sqrt{3}+\sqrt{6}+3\sqrt{2}+3}{3}$$

Note:

Fractional surds having a 'one term' expression in the denominator, e.g., $\frac{1}{\sqrt{2}}$, $\frac{3}{4\sqrt{6}}$ and $\frac{4}{9\sqrt{10}}$ are rationalised by **multiplying both numerator and** denominator by the denominator.

If the denominator is a 'two term' expression, e.g., $\frac{1}{\sqrt{2}+3}$, $\frac{4}{\sqrt{7}-\sqrt{5}}$ and $\frac{6}{\sqrt{x}-1}$

Then the fractional surd is rationalised by multiplying both numerator and denominator by the conjugate of the denominator.

Express the following in such a way that there is no surd in the denominator:

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{\sqrt{2}+1}{\sqrt{7}}$ (c) $\frac{1}{\sqrt{3}+1}$ (d) $\frac{\sqrt{2}}{\sqrt{11}-4\sqrt{10}}$

Exercise 4.2

Rationalise and simplify the denominators in numbers 1 – 4.

1. (a) $\frac{1}{\sqrt{2}}$ (c) $\frac{3}{\sqrt{3}}$ (c) $\frac{5}{\sqrt{11}}$

   (d) $\frac{1}{2\sqrt{10}}$ (e) $\frac{2}{3\sqrt{2}}$ (f) $\frac{3}{5\sqrt{3}}$

2. (a) $\frac{6}{3-\sqrt{3}}$ (b) $\frac{26}{4-\sqrt{3}}$ (c) $\frac{5}{3-\sqrt{2}}$ (d) $\frac{2\sqrt{3}}{7-\sqrt{5}}$

3. (a) $\frac{5\sqrt{7}}{\sqrt{5}-\sqrt{2}}$ (b) $\frac{2\sqrt{11}}{\sqrt{11}-\sqrt{5}}$ (c) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{7}-\sqrt{5}}$

   (d) $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{5}+\sqrt{7}}$ (e) $\frac{\sqrt{4}+1}{\sqrt{4}+\sqrt{5}}$

4. (a) $\frac{\sqrt{7}-2\sqrt{3}}{1+3\sqrt{2}}$ (b) $\frac{3\sqrt{2}-5\sqrt{11}}{4\sqrt{15}-2\sqrt{7}}$ (c) $\frac{\sqrt{3}-4}{\sqrt{4}-\sqrt{5}}$

   (d) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{6}}$ (e) $\frac{\sqrt{5}}{3\sqrt{15}-2\sqrt{10}}$

5. **Evaluate** the following, given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$ and $\sqrt{7} = 2.646$.

   (a) $\frac{1}{\sqrt{8}}$ (b) $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}$ (c) $\frac{3 + \sqrt{7}}{\sqrt{20}}$

   (d) $\frac{2\sqrt{3}}{\sqrt{5} + \sqrt{6}}$ (e) $\frac{\sqrt{5}}{\sqrt{7} - \sqrt{6}}$

6. **Given that** $a = \sqrt{5}$ and $b = \sqrt{7}$, express the following in terms of a and b **and simplify**:

   (a) $2\sqrt{5} - 6\sqrt{35}$ (b) $3\sqrt{5} + \sqrt{140}$

   (c) $\frac{\sqrt{20} + \sqrt{112}}{\sqrt{5} + \sqrt{28}}$ (d) $\sqrt{320} - \sqrt{7}$

7. **Simplify,** leaving your answer in surd form:

   (a) $\frac{1}{\sqrt{17} - 2\sqrt{5}} - \frac{1}{\sqrt{17} + 2\sqrt{5}}$ (b) $\frac{2}{\sqrt{6} + \sqrt{3}} - \frac{5}{\sqrt{7} - \sqrt{5}}$

8. Express in surd form and simplify by rationalising the denominator:

   (a) $\frac{2}{1 - \cos 45^0}$ (b) $\frac{\sqrt{10}}{1 - \tan 60^0}$ (c) $\frac{1 + \cos 30^0}{1 - \sin 60^0}$

9. **Express** as a surd the area of an equilateral triangle of sides 5 cm.

10. **Find** the longest side of right-angled triangle whose other sides are: $(\sqrt{2} + 1)$cm and $(\sqrt{2} - 1)$cm long.

11. Solve the equations:

    (a) $\sqrt{x + 5} = 5 - \sqrt{x}$ (b) $\sqrt{x} + \sqrt{x + 7} = 7$

Hint: square both sides.

Chapter Five

FURTHER LOGARITHMS

We have learnt that when a number is written in the form $C = a^b$, a is the base, while b is the index. This is the Index notation of a number. There is also logarithmic notation of a number. For example, a statement such as:
$25 = 5^2$ can be written as;
$\log_5 25 = 2$ and read as; the 'logarithm of 25 to base 5 is 2'. We therefore interpret the logarithm as the power to which 5 must be raised to give 25.

Consider the following cases:

(i) $2^3 = 8$

(ii) $3^2 = 9$

(iii) $\left(\frac{1}{5}\right)^{-2} = 25$

The three statements can be written in logarithmic notation as;

(a) $\log_2 8 = 3$

(b) $\log_3 9 = 2$

(c) $\log_{\frac{1}{5}} 25 = -2$

In general, if $a^b = c$, then $\log_a c = b$.

Note:
$a^0 = 1$
$\therefore \log_a 1 = 0$
Also, $a^1 = a$
$\therefore \log_a a = 1$
That is the logarithm of 1 to any base is zero and the logarithm of any number to the base equal to the number is 1.

Logarithms of numbers can be read directly from the tables of logarithms or by using a calculator, which gives answers to base 10.
From tables, $\log_{10} 5 = 0.6990$, which is written as $\log 5 = 0.6990$. (4 s.f.)
When using a calculator to get $\log_{10} 5$, the procedure is as follows:
Step I Key in 5.
Step II Press the log buttom.

The answer 0.6990 (to 4 s.f.) will be displayed on the screen.

Use a calculator to get logarithms of the following numbers:

(i) Log 4

(ii) Log 12

(iii) Log 20

Express the following in logarithmic notation:

(i) $5^3 = 125$

(ii) $8^4 = 4\,096$

(iii) $3^3 = 27$

(iv) $10^{-1} = 0.1$

6.11 Laws of Logarithms

We saw earlier that:

(i) $a^m \times a^n = a^{(m+n)}$

(ii) $a^m + a^n = a^{(m-n)}$

(iii) $(a^m)^n = a^{mn}$

Considering the above;

(i) $a^m \times a^n = a^{(m+n)}$

If $x = a^m$ and $y = a^n$, then;

$\log_a x = m$ and $\log_a y = n$ ....(1)

Therefore, $a^m \times a^n = xy$ ....(2)

$a^{(m+n)} = xy$

Hence, $\log_a xy = m + n$ ....(3)

But from equation (1) $\log_a x = m$ and $\log_a y = n$.

Substituting this in equation (3), we get;

$\log_a x = \log_a x + \log_a y$.... (4)

(b) $a^m + a^n = a^{(m-n)}$

If $x = a^m$ and $y = a^n$, then;

$\log_a x = m$ and $\log_a y = n$ ....(1)

Therefore, $a^m + a^n = x \div y$

$$\frac{a^m}{a^n} = \frac{x}{y}$$

$a^{(m-n)} = \frac{x}{y}$ ....(2)

Hence, $\log_a \frac{x}{y} = m - n$ ....(3)

In equation (1), $\log_a x = m$ and $\log_a y = n$.

Substituting in (3);

$\log_a(\frac{x}{y}) = \log_a x - \log_a y$ ....(4)

(c) Suppose $\log_a x^n = y$ ....(1)

In index notation, this can be written as;

$a^y = x^n$.

Finding the nth root on both sides;

$a^{\frac{y}{n}} = x$ ....(2)

Hence, $\log_a x = \frac{y}{n}$

Multiplying both sides by n;

$n \log_a x = y$ ....(3)

From equation (1) $\log x^n = y$

Substituting for y in equation (iii);

$n \log_a x = \log_a x^n$ or $\log_a x^n = n\log_a x$ .... (4)

Equation (4) in each of the above cases form the three laws of logarithms, stated as;

(i) $\log_a xy = \log_a x + \log_a y$

(ii) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(iii) $\log_a x^n = n \log_a x$

Example 1

Express the following in terms of log a, log b and log c.

(a) log ab (b) log abc (c) $\log \frac{a}{b}$ (d) $\log \frac{a}{bc}$

Solution

(a) log ab = log a + log b

(b) log abc = log a + log b + log c

(c) $\log \frac{a}{b}$ = log a – log b

(c) $\log \frac{a}{bc}$ = log a – (log b + log c)

= log a – log b – log c

Example 2

(a) Express in terms of log a, log b and log c;

$\log \left(\frac{a^2}{b^3}\right)$

(b) Write log 100 – 2 log 50 as a logarithm of a single number.

Solution

(a) $\log \left(\frac{a^2}{b^3}\right) = \log a^2 - \log b^3$

$= 2 \log a - 3 \log b$

(b)

$$\begin{aligned}\log 100 - 2\log 50 &= \log 100 - \log 50^2\\ &= \log\left(\frac{100}{50^2}\right)\\ &= \log\left(\frac{100}{2\,500}\right)\\ &= \log\frac{1}{25}\end{aligned}$$

Example 3

Given that log 2 = 0.3010, and log 3 = 0.4771, simplify:

(a) log 6 (b) log 1.5 (c) log 54

Solution

(a)

$$\begin{aligned}\log 6 &= \log(3 \times 2)\\ &= \log 3 + \log 2\\ &= 0.4771 + 0.3010\\ &= 0.7781\end{aligned}$$

(b)

$$\begin{aligned}\log 1.5 &= \log(3 \div 2)\\ &= \log 3 - \log 2\\ &= 0.4771 - 0.3010\\ &= 0.1761\end{aligned}$$

(c)

$$\begin{aligned}\log 54 &= \log(27 \times 2)\\ &= \log(3^3 \times 2)\\ &= 3\log 3 + \log 2\\ &= 3(0.4771) + 0.3010\\ &= 1.4313 + 0.3010\\ &= 1.7323\end{aligned}$$

Exercise 5.1

1. Express the following in logarithmic notation:

(a) $7^2 = 49$ (b) $4^{\frac{5}{2}} = 32$ (c) $1\,331 = 121^{\frac{3}{2}}$

(d) $10^{-2} = 0.01$ (e) $5^0 = 1$ (f) $8^{-\frac{1}{3}} = \frac{1}{2}$

(g) $125 = \left(\frac{1}{5}\right)^{-3}$ (h) $9^{-\frac{3}{2}} = \frac{1}{27}$ (i) $1 = a^0$

2. Express in index notation:

(a) $\log_5 625 = 4$ (b) $\log_{10} 1000 = 3$ (c) $\log_3 27 = 3$

(d) $\log 1 = 0$ (e) $\log_{\frac{1}{2}} 4 = -2$ (f) $\log_{25} 5 = \frac{1}{2}$

(g) $\log_x y = 2$

3. Express in terms of a, b and c, given that $\log x = a$, $\log y = b$ and $\log z = c$:

   (a) $\log \frac{1}{x}$ (b) $\log x^4y$ (c) $\log \left(\frac{x}{y}\right)$

   (d) $\log \left(\frac{x^2}{y}\right)$ (e) $\log \left(\frac{z^2}{10}\right)$ (f) $\log \frac{1}{10x}$

   (g) $\log \frac{100}{\sqrt{z}}$

4. Simplify:

   (a) $\log 3 + \log 4$ (b) $\log 6 - \text{Log } 2$

   (c) $\log 30 - \log 3$ (d) $\log 2 + \log 6 - \log 4$

   (e) $\frac{1}{2}\log 4 - \log 6$ (f) $2 - 2 \log 5$

   (g) $\log_2 16 - \log_2 8$ (h) $\log_3 2.7 + \log_3 10$

5. Simplify:

   (a) $\log (x + 1) - \log (x^2 - 1)$ (b) $2 \log_a 5 + \log_a 4 - 2 \log_a 10$

   (c) $\log x^4 + \log x$ (d) $\log_9 3 + \log_9 27$

   (e) $\log 3 + \log 15 - \log 4.5$ (f) $\log_8 72 - \log_8 \frac{9}{8}$

6. Given that $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find:

   (a) $\log 12$ (b) $\log 5$ (c) $\log 108$

5.2: Logarithmic Equations and Expressions

Consider the following equations;

(i) $\log_3 81 = x$ (ii) $\log_x 8 = 3$

The value of x in each case is established as below:

(i) $\log_3 81 = x$

$\therefore 3^x = 81$

$3^x = 3^4$

$x = 4$

(ii) $\log_x 8 = 3$

$x^3 = 8$

$x^3 = 2^3$

$x = 2$

Example 4

Solve for x in $\log x + \log 5 = \log 30$.

Solution

$\log x + \log 5 = \log 30$

$\therefore\ \log x = \log 30 - \log 5$

$\log x\ = \log\left(\frac{30}{5}\right)$

$\log x = \log 6$

$\therefore\ x = 6$

or

$\log x + \log 5 = \log 30$

$\therefore\ \log 5x = \log 30$

$5x = 30$

$x = 6$

Example 5

Find y without using tables if $2 + \log_2 3 + \log_2 y = \log_2 5 + 1$

Solution

$2 + \log_2 3 + \log_2 y\ =\ \log_2 5 + 1$

$-1 + 2 + \log_2 3 + \log_2 y\ =\ \log_2 5$

$1 + \log_2 3 + \log_2 y\ =\ \log_2 5$

But $1 = \log_2 2$

Substituting;

$\log_2 2 + \log_2 3 + \log_2 y\ =\ \log_2 5$

$\log_2(2 \text{ x } 3)^2 + \log_2 y = \log_2 5$

$\log_2 6 + \log_2 y = \log_2 5$

$\log_2 y = \log_2 5 - \log_2 6$

$\log_2 y\ =\ \log_2\left(\frac{5}{6}\right)$

$\therefore\ y\ =\ \frac{5}{6}$

Example 6

Simplify without using tables;

$$\frac{\log 25 + \log 625}{\log 5}$$

Solution

$$\frac{\log 25 + \log 625}{\log 5} = \frac{\log 5^2\ + \log 5^4}{\log 5}$$

$$= \frac{2 \log 5 + 4 \log 5}{\log 5}$$

$$= \frac{6 \log 5}{\log 5}$$

$$= 6$$

Simplify without using tables:

(i) $\frac{\log 8}{\log 2}$ (ii) $\frac{\log x^4}{\log x}$ (iii) $\frac{\log \sqrt{6}}{\log 6}$

5.3: Further Computation using Logarithms

Consider the expression $\log_8 3$

The solution to $\log_8 3$ cannot be read from tables, which give logarithms to base 10 only. However, if we let;

$\log_8 3 = x$, then $8^x = 3$.

To solve for x, we introduce logarithms to base 10 on both sides. That is;

$\log 8^x = \log 3$

$x \log 8 = \log 3$

$$x = \frac{\log 3}{\log 8}$$

From tables the logarithms of 3 and 8 are 0.4771 and 0.9031 respectively. Substituting this is;

$$x = \frac{0.4771}{0.9031}$$

$x = 0.5283$ (4 s.f.)

$\therefore \log_8 3 = 0.5283$.

Example 7

Solve $\log_6 2$

Solution

Let $\log_6 2 = t$. Then, $6^t = 2$

Introducing logarithms to base 10 on both sides;

$\log 6^t = \log 2$

$t \log 6 = \log 2$

$$t = \frac{\log 2}{\log 6}$$

$$t = \frac{0.3010}{0.7782}$$

$t = 0.3868$

$\therefore \log_6 2 = 0.3868$

Consider the equation $2^{2x} + 3(2^x) - 4 = 0$.

Taking logs on both sides cannot assist in getting the value of x, since $2^{2x} + 3(2^x)$ cannot be combined into a single expression. However, if we let $2^x = y$, then the equation becomes quadratic in y.

Thus, let $2^x = y$ ....(1)
$\therefore\ y^2 + 3y - 4 = 0.$ ....(2)
$(y + 4)(y - 1) = 0$
$y = -4$ or $y = 1$

Substituting for y in equation (1);
$2^x = -4$ or $2^x = 1$
There is no real value of x for which $2^x = -4$.
Hence, $2^x = 1$
$x = 0$

Example 8
Solve for x in $(\log_{10}x)^2 = 3 - \log_{10}x^2$

Solution
$(\log_{10}x)^2 = 3 - 2\log_{10}x$
Let $\log_{10}x = t$ ....(1)
$\therefore\ t^2 = 3 - 2t$
$t^2 + 2t - 3 = 0$
$t^2 + 3t - t - 3 = 0$
$t(t + 3) - 1(t + 3) = 0$
$(t - 1)(t + 3) = 0$
$t = 1$ or $t = -3$
Substituting for t in equation (1).
$\log_{10}x = 1$ or $\log_{10}x = -3$
$\therefore\ 10^1 = x$ or $10^{-3} = x$
$x = 10$ or $\frac{1}{1\,000}$

Note:

$$\log_b a = \frac{1}{\log_a b}$$

Exercise 5.2
Simplify:

1. (a) $\dfrac{\log 125}{\log 5}$ (b) $\log_a a^2 \div \log_a \sqrt{a}$ (c) $(\log a^2 + \log b^2) - \log ab$
2. Evaluate:
   (a) $\log_3 4$ (b) $\log_7 17$ (c) $\log_5 14$ (d) $\log_6 28$
   (e) $\log_{12} 41$ (f) $\log_2 10$ (g) $\log_7 5$

3. Find the value of x in the equation:
   (a) $4^x = 3$ (b) $7^x = 11$ (c) $10^x = 7$
   (d) $6^x = 15$ (e) $2^{2x} = 5$ (f) $3^{(x-1)} = 7$
   (g) $(5^x)\, 5^{(x-1)} = 10$ (h) $2^{x+1} = 3^x$
4. Express y in terms of x in the equation:
   $\log_6 x + \log_6 y^3 = 1.$
5. If $\log_2 x + \log_x 2 = 2$, find x.
6. Solve the simultaneous equations below:
   (a) $\log_x y = 2$
   $xy = 8$
   (b) $2 \log y = \log 2 + \log x$
   $2^y = 4^x$
   (c) $\text{Log}\,(x + y) = 0$
   $2 \log x = \log (y + 1).$
7. Solve the following:
   (a) $\log x = 4 \log 2$ (b) $\log 12 + 3\log x = \log 96$
   (c) $1 + \log_5 x = \log_5 12$ (d) $\log_3 y - 4 = \log_3 5$
   (e) $\log_4 P^2 = 2 \log_4 5$ (f) $\log_3 4 + \log_3 x + \log 3 - \log_3 5 = \log_3 2$
   (g) $\log_2 y = \log_2 3 + \log_2 7 + 2\log_2 y$
   (h) $2^2 + \log_2 x^2 + 5\log_2 2 = 9$
8. Solve for x in $\log(3x + 4) - \log(3 - x) = 1$

Chapter Six

COMMERCIAL ARITHMETIC (II)

6.1: Simple Interest

Interest is the money charged for the use of borrowed money for a specific period of time. If a sum of money is deposited in, or borrowed from a financial institution for a period of time, it earns interest. The sum of money borrowed or deposited is called the **principal (P)**. The ratio of interest earned in a given period of time to the principal is called the **rate (R) of interest.** The rate of interest is normally expressed as a percentage of the principal per annum (p.a.)

When interest is calculated using only the initial principal at a given rate and time, it is called **simple interest (I)**.

Example 1

Find the simple interest earned on sh. 2 000 at 12% per annum for:

(a) 1 year. (b) 2 years. (c) 3 years.

Solution

(a) $I = \text{sh. } \frac{12}{100} \times 2\,000 \times 1$

$= \text{sh. } 240$

(b) $I = \text{sh. } \frac{12}{100} \times 2\,000 \times 2$

$= \text{sh. } 480$

(c) $I = \text{sh. } \frac{12}{100} \times 2\,000 \times 3$

$= \text{sh. } 720$

We notice that in simple interest situations, the interest earned is constant for every year. In the example above, the interest earned in the first year is sh. 240, in the second year an additional sh. 240, and so on.

Thus if sh. P is borrowed at R% per annum for T years, then, interest for 1 year sh. $\frac{R}{100} \times P \times 1$ is given by;

For T years it is $= \frac{R \times P \times T}{100}$

This can be abbreviated as $I = \frac{PRT}{100}$

This is the **simple interest formula**. The sum of money at the end of a given period of time is referred to as the **amount** (A).

Thus, Amount (A) = principal (P) + interest (I)
$A = P + I$

Example 2

Calculate the simple interest and the amount on sh. 16 000 for $1\frac{1}{2}$ years at 14% per annum.

Solution

$I = \frac{PRT}{100}$
$= \text{sh. } 16\,000 \times \frac{14}{100} \times \frac{3}{2}$
$= \text{sh. } 3\,360$
Amount $= P + I$
$= \text{sh. } 16\,000 + \text{sh. } 3\,360$
$= \text{sh. } 19\,360$

Example 3

Calculate the rate of interest if sh. 4 500 earns sh. 500 after $1\frac{1}{2}$ years.

Solution
From the simple interest formula;
$I = \frac{PRT}{100}, R = \frac{100\,I}{PT}$
P = sh. 4 500
I = sh. 500
$T = 1\frac{1}{2}$ years

$$\text{Therefore, } R = \frac{100 \times 500}{4\,500 \times 1\frac{1}{2}}$$
$$= 7.4\%$$

Example 4
Juma invested a certain amount of money in a bank which paid 12% p.a. simple interest. After 5 years, his total savings were sh. 5 600. Determine the amount of money he invested initially.

Solution
Let the amount invested be sh. P
T = 5 years

R = 12% p.a.
A = sh. 5 600
But A = P + I

Therefore 5 600 = $P + P \times \frac{12}{100} \times 5$
$= P + 0.60\,P$
$= 1.6\,P$

$\therefore P = \frac{5\,600}{1.6}$
= sh. 3 500

Exercise 6.1

1. Calculate the simple interest on each of the following:
   (a) sh. 15 300 at $11\frac{1}{2}$% p.a. for 3 years.
   (b) sh. 15 600 at 13% p.a. for 4 years.
   (c) sh. 23 700 at $12\frac{1}{4}$% p.a. for 5 years.
   (d) sh. 18 000 at 9% p.a. for $1\frac{1}{2}$ years.
   (e) sh. 15 000 at 10% p.a. for 9 months.
   (f) sh. 7 800 at $13\frac{1}{2}$% p.a. for 17 months.
2. Calculate the rate of interest in each of the following:
   (a) sh. 48 500 earning sh. 7 500 in $2\frac{1}{2}$ years.
   (b) sh. 12 000 earning sh. 4 000 in 2 years.
   (c) sh. 13 200 earning sh. 3 300 in $2\frac{1}{2}$ years.
   (d) sh. 20 400 earning sh. 2 975 in $2\frac{1}{3}$ years.
   (e) sh. 4 760 earning sh. 1 870 in $1\frac{1}{2}$ years.
   (f) sh. 16 400 earning sh. 3 280 in 16 months.
3. Find the time during which simple interest on:
   (a) sh. 7 200 at $8\frac{1}{2}$% p.a. accrues to sh. 1 224.
   (b) sh. 24 800 at $6\frac{1}{4}$% p.a. accrues to sh. 4 650.
   (c) sh. 9 500 at 5% p.a. accrues to sh. 750.
   (d) sh. 20 000 at $12\frac{1}{2}$% p.a. accrues to sh. 4 000.
   (e) sh. 50 000 at 10% p.a. accrues to sh. 12 500.
   (f) sh. 14 700 at $2\frac{2}{9}$% p.a. accrues to sh. 3 675.

4. Calculate the principal that will earn the following simple interests:
   - (a) sh. 7 200 in 2 years at 12% p.a.
   - (b) sh. 1 050 in 6 months at 15% p.a.
   - (c) sh. 2 000 in 4 months at $12\frac{1}{2}$% p.a.
   - (d) sh. 11 200 in 3 years at $9\frac{1}{3}$% p.a.
   - (e) sh. 16 000 in 2 years at 15% p.a.
   - (f) sh 3 629.50 in 10 years at 16% p.a.
5. Calculate the amount earned on each of the following:
   - (a) sh. 12 000 for 3 years at 12% p.a.
   - (b) sh. 50 000 for $1\frac{1}{2}$ years at $12\frac{1}{2}$% p.a.
   - (c) sh. 150 000 for 5 years at 14% p.a.
   - (d) sh. 450 000 for 5 years at $7\frac{1}{2}$% p.a.
   - (e) sh. 60 000 for 12 months at 14% p.a.
   - (f) sh 8 670 for $\frac{1}{4}$ year at $13\frac{1}{3}$% p.a.
6. Find the simple interest that would accrue when a welfare group lends to a member sh. 15 000 at the rate of 12% in one year.
7. If a customer deposits sh. 26 000 in a commercial bank which pays simple interest at the rate of 14% p.a., find the amount after 3 years.
8. The simple interest on a given sum of money borrowed for 4 years at 10% p.a. exceeds the simple interest on the same sum borrowed for $2\frac{1}{2}$ years at 12% p.a. by sh. 12 960. What was the sum of money borrowed?
9. A man deposited some money in a savings bank for $2\frac{3}{4}$ years and found that the money had earned sh. 8 600 simple interest. If the rate was $8\frac{1}{2}$% p.a., how much money did he have in his account at the end of the period?
10. A farmer borrowed sh. 12 460 from a financial institution. The simple interest rate was $12\frac{1}{2}$% p.a. After 6 months, he paid back sh. 8 460. How much did he still owe the bank including interest?
11. A butcher obtained a loan on which simple interest was charged at 14% p.a. He cleared his loan by paying sh. 24 805 at the end of $1\frac{1}{2}$ years. Find the sum borrowed.
12. A sum of sh. 7 600 was partly lent at 10% p.a. simple interest and partly at $12\frac{1}{2}$% p.a. simple interest. The total interest after 2 years was sh. 1 672. How much money was lent at $12\frac{1}{2}$%?

6.2: Compound Interest

If money is borrowed from or deposited in a financial institution, it earns interest after a specified period of time at a stated rate of interest. Instead of this interest being paid back to the owner, it may be added to (compounded with) the principal and thereafter also earns interest. Such interest is called **compound interest.** The period after which it is compounded to the principal is referred to as the **interest period.**

The compound interest may be calculated annually, semi-annually, quarterly, monthly, etc.

If the rate of compound interest is R% p.a. and the interest is calculated n times per year, then the rate of interest per period is $\left(\frac{R}{n}\right)$%. For example a rate of 20% p.a. compounded quarterly would give a rate of (20 ÷ 4)% = 5% per interest period.

After every interest period, the principal increases and therefore the interest earned also increases. The sum of the principal and the compound interest is called the **accumulated amount** (A).

Example 5

Find the amount after 2 years if sh. 10 000 is invested at 15% p.a. compound interest.

Solution

1st year interest = sh. $\frac{15}{100} \times 10\,000 \times 1$

= sh. 1 500

Amount after 1st year = sh.(10 000 + 1 500)

= sh. 11 500

2nd year interest = sh. $\frac{15}{100} \times 11\,500$

= sh. 1 725

Amount after 2nd year = sh. (11 500 + 1 725)

= sh. 13 225

Example 6

A customer deposited sh. 14 000 in a savings account. Find the accumulated amount after one year if interest was paid at 12% p.a. compounded quarterly.

Solution

Since in one year there are 4 interest periods, the rate per period is $12 \div 4 = 3\%$.

1st interest period:

Interest = sh. $\frac{3}{100} \times 14\,000 \times 1$

= sh. 420

Amount after 1st period = sh. (14 000 + 420)

= sh. 14 420

2nd interest period:

Interest = sh. $\frac{3}{100} \times 14\,420 \times 1$

= sh. 432.60

Amount after 2nd period = sh.(14 420 + 432.60)

= sh. 14 852.60

3rd interest period:

Interest = sh. $\frac{3}{100} \times 14\,852.60 \times 1$

= sh. 445.60

Amount after 3rd period = sh. (14 852.60 + 445.60)

= sh. 15 298.20

4th interest period

Interest = $\frac{3}{100} \times 15\,298.20$

= sh. 445.95

Amount after 4th period = sh. (15 298.20 + 458.95)

= sh. 15 757.15

∴ The accumulated amount is sh. 15 757.15.

***Note*:**

The interest period is taken to be annual, unless otherwise stated.

Compound Interest Formula

Suppose sh. P is invested at r% per period compound interest. The accumulated amount after 4 interest periods can be calculated as follows:

Amount after 1st period = sh. $(P + \frac{rP}{100})$

= sh. $P(1 + \frac{r}{100})$

2nd period:

Interest = sh. $\frac{r}{100} \times P(1 + \frac{r}{100})$

Amount = sh. $P(1 + \frac{r}{100})$ + sh. $\frac{r}{100} \times P(1 + \frac{r}{100})$

Factoring out $P(1 + \frac{r}{100})$;

$P(1 + \frac{r}{100})\{1 + \frac{r}{100}\} = P(1 + \frac{r}{100})^2$

3rd period:

Interest = sh. $\frac{r}{100} \times P(1 + \frac{r}{100})^2$

Amount = $P(1 + \frac{r}{100})^2 + \frac{r}{100} \times P(1 + \frac{r}{100})^2$

Factoring out $P(1 + \frac{r}{100})^2$;

$P(1 + \frac{r}{100})^2 (1 + \frac{r}{100}) = P(1 + \frac{r}{100})^3$

4th period:

Interest = $\frac{r}{100} \times P(1 + \frac{r}{100})^3$

$$\begin{aligned}\text{Amount} &= P(1 + \tfrac{r}{100})^3 + \tfrac{r}{100} \times P(1 + \tfrac{r}{100})^3 \\ &= P(1 + \tfrac{r}{100})^3(1 + \tfrac{r}{100}) \\ &= P(1 + \tfrac{r}{100})^4\end{aligned}$$

We can notice that for n interest periods, the accumulated amount would be

$A = P(1 + \frac{r}{100})^n$

This is the **compound interest formula.**

Example 7

Find the amount at the end of the fourth year if sh. 30 000 is deposited at 15% p.a. compound interest.

Solution

The interest period is 1 year, the rate per period (r) is 15% and the number of periods is 4.

$$\begin{aligned}\text{Amount at the end of the fourth year} &= \text{sh. } 30\,000(1 + \tfrac{15}{100})^4 \\ &= \text{sh. } 30\,000 \times 1.15^4 \\ &= \text{sh. } 52\,470\end{aligned}$$

Example 8

What would sh. 15 000 amount to after 3 years at 16% p.a. compounded quarterly?

Solution

For every year, we have 4 interest periods. Thus, in 3 years we will have 12 interest periods, i.e., $n = 12$

Rate per period $(r) = 16 \div 4 = 4\%$

$P =$ sh. 15 000

$$\therefore A = \text{sh. } 15\,000(1 + \tfrac{4}{100})^{12}$$
$$= \text{sh. } 15\,000 \times 1.04^{12}$$
$$= \text{sh. } 24015.50$$

Example 9

Find the accumulated amount if sh. 20 000 is deposited for $3\frac{1}{3}$ years at 10% p.a. compound interest.

Solution

$$\text{Amount after 3 years} = \text{sh. } 20\,000(1 + \tfrac{10}{100})^3$$
$$= \text{sh. } 20\,000 \times 1.1^3$$
$$= \text{sh. } 26\,620$$

$$\text{Interest for the last } \tfrac{1}{3} \text{ of the year} = \text{sh. } 26\,620 \times \tfrac{10}{100} \times \tfrac{1}{3}$$
$$= \text{sh. } 887.35$$

$$\therefore \text{Amount after } 3\tfrac{1}{3} \text{ years} = \text{sh. } (26\,620 + 887.35)$$
$$= \text{sh. } 27\,507.35$$

Example 10

A man wants to have sh. 30 000 in 8 years. How much money must he invest now if the rate of interest is 7.5% p.a. compound interest?

Solution

Using the compound interest formula;

$A =$ sh. 30 000

$r = 7.5\%$ per period

$n = 8$ interest periods

Therefore, $P(1 + \frac{7.5}{100})^8 =$ sh. 30 000

$P = \text{sh. } \frac{30\,000}{(1.075)^8}$

$= \text{sh. } 16\,821$

Example 11

Find the rate per annum at which a certain amount of money doubles after being invested for a period of 5 years compounded annually.

Solution

Let the amount invested be sh. P and the interest rate per annum be R%:
Amount after 5 years = sh. 2P

So, $2P = P(1 + \frac{R}{100})^5$

$2 = (1 + \frac{R}{100})^5$

Taking 5th roots of both sides

$(1 + \frac{R}{100}) = 1.149$ (logarithms or a calculator can be used to evaluate $\sqrt[5]{2}$)

$1 + \frac{R}{100} = 1.149$

$\frac{R}{100} = 0.149$

$R = 14.9\%$

Exercise 6.2

1. Find the compound interest in each of the following:
   (a) sh. 8 000 invested for 2 years at 13% p.a.
   (b) sh. 15 000 invested for $3\frac{1}{2}$ years at 14% p.a.
   (c) sh. 10 000 invested for 3 years at $8\frac{1}{4}$ p.a.
   (d) sh. 110 000 invested for 5 years at 14% p.a.
   (e) sh. 20 150 invested for 6 years at $5\frac{1}{2}$% p.a.
   (f) sh. 1 000 000 invested for 4 years at 15% p.a.
2. Find the accumulated amount for each of the following:
   (a) sh. 23 500 invested for 3 years at 10% p.a.
   (b) sh. 8 500 invested for 2 years at 5% p.a. compounded semi-annually.
   (c) sh. 17 250 invested for $3\frac{1}{2}$ years at 8% p.a.
   (d) sh. 15 000 invested for 2 years at 12% p.a. compounded quarterly.
   (e) sh. 10 000 invested for 2 years at 15% p.a. compounded monthly.
   (f) sh. 550 000 invested for $5\frac{1}{2}$ years at 11% p.a.

3. Find the time taken for each of the principals below to accumulate to the given amounts at the given rates of compound interest:
   (a) accumulated amount of sh. 112 896 from a principal of sh. 90 000 at a rate of 10% p.a.
   (b) accumulated amount of sh. 18 816 from a principal of sh. 15 000 at 12% p.a.
   (c) accumulated amount of sh. 314 175 from a principal of sh. 200 000 at 15% p.a.
   (d) accumulated amount of sh. 71 105 from a principal of sh. 50 000 at 18% p.a. compounded quarterly.
   (e) accumulated amount of sh. 17 908 from a principal of sh. 10 000 at a rate of 12% p.a. compounded semi-annually.
4. A farmer invested sh. 48 000 for 5 years. Calculate the rate per annum of compound interest if the accumulated amount was sh. 77 304.50.
5. A farmer sold his cows for sh. 25 000 and deposited this money in a savings account which paid 12% p.a. compound interest. After 2 years, he withdrew sh. 25 000 and left the rest for a further 2 years. What was the amount in the bank after the two periods?
6. A jua kali mechanic wants to accumulate sh. 400 000 in 5 years. How much money must he invest now at 10% p.a. compound interest?
7. A man sold a plot of land for sh. 80 000 and invested the money in a building society which pays 12% p.a. compounded semi-annually. After 2 years, he withdrew sh. 50 000 and left the rest for a further 3 years. Calculate the total interest he earned in the 5-year period.
8. How much more interest does sh. 10 000 earn after 5 years invested at 5% p.a. compound interest than if it is invested for the same period and rate at simple interest?
9. A businessman borrowed £1 500 at a rate of 9% p.a. compound interest. He paid £450 at the end of each year for 2 years. How much did he have to pay at the end of the third year in order to settle his debt?
10. (a) What rate of interest (p.a.) is equivalent to:
      (i) 1% per month.
      (ii) 3% every 3 months.
      (iii) 6% every 6 months.
    (b) What rate of interest p.a. compounded semi-annually is equivalent to 10% p.a. compounded quartely?
    (c) What rate of interest p.a. compounded semi-annually is equivalent to 12% p.a. compounded monthly?

6.3: Appreciation and Depreciation

Cars and clothes in use experience wear and tear with time. Consequently, they lose their value. The loss in value of an asset with time is called **depreciation**. If, however, an asset gains in value with time, it is said to have **appreciated**. For example the value of land increases with time due to high demand.

Example 12

An iron box costs sh. 500. Every year it depreciates by 10% of its value at the beginning of that year. What will its value be after 4 years?

Solution

Value after the first year = sh. $(500 - \frac{10}{100} \times 500)$

= sh. 450

Value after the second year = sh.$(450 - \frac{10}{100} \times 450)$

= sh. 405

Value after the third year = sh. $(405 - \frac{10}{100} \times 405)$

Value after the fourth year = sh.$(364.50 - \frac{10}{100} \times 364.50)$

= sh. 328.05

Show that the value of the iron box after the four years is $500(1 - \frac{10}{100})^4$.

In general, if P is the initial value of an asset, A the value after depreciation for n periods and r the rate of depreciation per period;

$$A = P(1 - \frac{r}{100})^n$$

Example 13

A minibus costs sh. 400 000. Due to wear and tear, it depreciates in value by 2% every month. Find its value after one year.

Solution

$$A = P(1 - \frac{r}{100})^n$$

Substituting P = 400 000, r = 2, and n = 12 in the formula;

A = sh. $400\,000(1 - 0.02)^{12}$

= sh. $400\,000(0.98)^{12}$

No.	Log.
0.98^{12}	$\bar{1}.9912 \times 12$
	$= \bar{1}.8944$
400 000	5.6021
3.137×10^5	5.4965

Therefore, A = sh. 313 700

Example 14

The initial cost of a ranch is sh. 5 000 000. At the end of each year, the land value increases by 2%. What will be the value of the ranch at the end of 3 years?

Solution

Value after 1st year $=$ sh.$(5\ 000\ 000 + \frac{2}{100} \times 5\ 000\ 000)$
$=$ sh. 5 100 000
Value after 2nd year $=$ sh.$(5\ 100\ 000 + \frac{2}{100} \times 5\ 100\ 000)$
$=$ sh. 5 202 000
Value after 3rd year $=$ sh.$(5\ 202\ 000 + \frac{2}{100} \times 5\ 202\ 000)$
$=$ sh. 5 306 040

Alternatively;

The value of the ranch after 3 years $=$ sh. $5\ 000\ 000(1 + \frac{2}{100})^3$
$=$ sh. $5\ 000\ 000(1.02)^3$
$=$ sh. 5 306 040.

Therefore the value of the ranch after the 3 years is sh. 5 306 040.

In general, if P is the initial value of an asset, A the value after appreciation for n periods and r the appreciation rate per period;

$A = P(1 + \frac{r}{100})^n$

Note:

With a slight change of meaning to A and P in the formulae for appreciation and depreciation, they can be used to solve problems involving situations where there is a uniform rate of change over a period of time, e.g., growth of animal or town populations.

Exercise 6.3

1. A school van was bought for K£75 000. Each year, its value decreases by 10%. Find its value after the second and third year.
2. A piece of land was valued at sh. 150 000. The value of the land appreciates every year by 12%. What will be the value of the land after 4 years?
3. A coffee hullery depreciates in value at the rate of 8% p.a. If the original cost of the hullery was sh. 98 000, find its value after 5 years.
4. The population of a town increases at 4% p.a. If the present population of the town is 500 000, what will it be after 5 years?
5. A car which costs sh. 200 000 depreciates in value at the rate of 5% every 6 months. Find its value after 4 years.
6. The cost of an article is sh. 1 200. Find the rate of inflation if the cost of the article after 5 years is sh. 1 932.60.
7. A seedling is 4 cm high. If its average growth rate is 20% per week, how long will it take for it to be 9.953 cm high?
8. At the end of the year 2000, the population of a centre was 30 000. What was the population of this centre at the beginning of 1997 if the growth rate was 3% per year?
9. What was the population of a town A four years ago if the present population is 800 000 and the growth rate is 5% p.a.?
10. The area covered by a desert is 40 000 km^2 at present. If the desertification rate is estimated to be 2% every 10 years, calculate the area of the desert in 30 years time.
11. Statistics from a town in Kenya indicate that 160 000 people are HIV positive. It has also been established that the rate of its spread in this town is 8% p.a. How many people will be carriers in 6 years time?
12. A commercial plot was valued at sh. 80 000. The plot appreciates in value at 20% in the first year, 15% in the second year and 10% in the third year. Find the value of the plot at the end of the third year.
13. The present cost of a ranch is sh. 4 072 000. It cost 10 years ago was sh. 2 500 000. If this ranch has been appreciating in value uniformly, determine the annual rate of appreciation.
14. A building was valued at sh. 500 000 in January 1999. Due to high demand, the building appreciated by 15% every year. What was its value at the end of December 2001?

6.4: Hire Purchase

Many a time, individuals find themselves compelled to spend huge sums of money on basic needs and emergencies. As a result, they have to acquire certain items on credit. To buy on credit implies making a down payment (deposit) then paying the balance by installments (usually regular).

In this method, interest has to be against the borrowing. The rate of interest charged depends on the time, the amount and the lending body.

Credit traders, allow customers to buy such items as TV sets, sewing machines, fridges, cookers etc., this way. The following examples illustrate how the loans are repaid.

Example 15

Achieng' wants to buy a sewing machine on hire purchase. It has a cash price of sh. 7 500. She can pay the cash price or make a down payment of sh. 2 250 and 15 monthly instalments of sh. 550 each. How much interest does she pay under the instalment plan?

Solution

Total amount of instalments = sh. 550 × 15
= sh. 8 250
Down payment (deposit) = sh. 2 250
Total payment = sh. (8 250 + 2 250)
= sh. 10 500
Amount of interest charged = sh.(10 500 – 7 500)
= sh. 3 000

Note that Achieng' takes the sewing machine as soon as she pays the down payment.

The interest charged for buying goods/services on credit is called the **carrying charge** and is calculated as a compound interest.This method of buying goods/services by instalments is called **hire purchase.**

Example 16

Calculate the rate of interest charged per month in example 15.

Solution

When a down payment is made, the compound interest is charged on the balance from the cash price of the given item.

Here, amount borrowed = sh. (7 500 – 2 250)
= sh. 5 250 (this is the new principal)

Let the rate of interest be r% p.m.

Then, $A = P(1 + \frac{r}{100})^n$

A is the principal and the carrying charge.

P – is the money borrowed (new principal). Where no deposit is paid this is the same as the cash price.

Thus, $8\ 250 = 5\ 250(1 + \frac{r}{100})^{15}$

Therefore, r = 3.059%

Exercise 6.4

1. The cash price of a bed is sh. 6 500. A man buys it on hire purchase by making a down payment of sh. 2 300 followed by 12 monthly instalments of sh. 490. Calculate the carrying charge.
2. The hire purchase price of an iron box is sh. 840. If a down payment of sh. 160 is made and the balance paid in 10 equal monthly instalments, calculate the size of each instalment.
3. The cash price of a cooker is sh. 9 000. A customer bought the cooker by paying 15 monthly instalments of sh. 950 each. Calculate:
   (a) the carrying charge.
   (b) the rate of interest.
4. A bicycle can be bought for sh. 3 500 cash or by paying 10 monthly instalments of sh. 490 each. Find the carrying charge and the rate of interest charged per annum.
5. A colour television whose cash price is sh. 30 000 can be sold by two methods:
   (a) 15 monthly instalments of sh. 2 350 each.
   (b) Paying $\frac{1}{4}$ of the cash price as deposit and 14 monthly instalments of sh. 1 875 each.
   Which of the two methods would be cheaper, and by how much?
6. A businessman offers a photocopying machine for either a down payment of sh. 5 000 and 15 monthly instalments of sh. 1 050 each, or a down payment of sh. 7 500 and 10 monthly instalments of sh. 1 250 each.
   (a) Which plan is cheaper and by how much?
   (b) If the machine can be bought for sh. 18 000 cash, which of the two plans carries a higher rate of interest?

6.5: Income Tax

A person who is employed receives payments in some form for work done. Such payments may include salaries, housing allowances, medical allowances, hardship allowances, etc.

Each person getting such payments is expected to pay back part of it to the government in the form of taxes. Such a tax on personal income is known as **income tax**.

An individual's **gross income** is the total amount of money due to the individual at the end of the month or year. This includes the salary and any other allowances or benefits due to the individual.

Taxable income is the amount on which tax is levied. This is the gross income less any special benefits on which taxes are not levied. Such benefits include refunds for expenses incurred while one is on official duty.

In order to calculate the income tax that one has to pay, we convert the taxable income into Kenya pounds (K£) per annum or per month as dictated by the table of rates given. We then divide the taxable income into a slab of K£ 5 808 and the rest in slabs of K£ 5 472 as the table below starting from the lowest rate. The remainder is taxed at the next tax bracket.

The rates of income tax are given by the government and keep on changing from time to time. The table below gives the rate applicable in Kenya in the year 2003.

Table 6.1

Income (K£ per annum)	*Rate (sh. per pound)*
1 – 5 808	2
5 809 – 11 280	3
11 281 – 16 752	4
16 753 – 22 224	5
Excess over 22 224	6

Relief

(i) Every employee in Kenya is entitled to an automatic personal tax relief of sh. 12 672 p.a. (sh. 1 056 per month).

(ii) An employee with a life insurance policy on his life, that of his wife or child, may make a tax claim on the premiums paid towards the policy at sh. 3 per pound subject to a maximum claim of sh. 3 000 per month.

Example 17

Mr Langat earns a total of K£12 300 p.a. Calculate how much tax he should pay per annum.

Solution

Langat's income lies between £1 and £12 300. The highest tax band is therefore the third band.

For the first £5 808, tax due is sh. 5 808 × 2 = sh. 11 616
For the next £5 472, tax due is sh. 5 472 × 3 = sh. 16 416
Remaining £1 020, tax due sh. 1 020 × 4 = sh. 4 080 +
Total tax due sh. 32 112
Less personal relief of sh. 1 056 × 12 = sh. 12 672 –
sh. 19 440

∴ Tax payable p.a. is sh. 19 440.

Example 18

Mr. Masimbwa earns a basic salary of sh. 15 000 per month. In addition, he gets a medical allowance of sh. 2 400 and a house allowance of sh. 12 000. Use the tax brackets in table 6.1 to calculate the tax he pays in a year.

Solution

Taxable income per month = sh.(15 000 + 2 400 + 12 000)
= sh. 29 400

Converting to K£ p.a. = K£ 29 400 × $\frac{12}{20}$
= K£17 640

Tax due
First £5 808 = sh. 5 808 × 2 = sh. 11 616
Next £5 472 = sh. 5 472 × 3 = sh. 16 416
Next £5 472 = sh. 5 472 × 4 = sh. 21 888
Remaining £888 = sh. 888 × 5 = sh. 4 440 +
Total tax due sh. 54 360
Less personal relief sh. 12 672 –
Therefore, tax payable p.a. sh. 41 688

PAYE

In Kenya, every employer is required by law to deduct income tax from the monthly earnings of his employees every month and to remit the money to the income tax department. This system is called Pay As You Earn (PAYE).

To assist employers, PAYE tables are published by the income tax department showing the various levels of monthly income, in pounds and the corresponding tax payable.

Housing

If an employee is provided with a house by the employer (either freely or for a nominal rent) then 15% of his salary is added to his salary (less rent paid) for purposes of tax calculation. If the tax payer is a director and is provided with a free house, then 15% of his basic salary is added to his salary before taxation.

Example 19

Mr. Mwashighadi who is a civil servant lives in a government house for which he pays a rent of sh. 500 per month. If his salary is £9 000 p.a., calculate how much PAYE he remits monthly.

Solution

Basic salary		£9 000
Housing £$\frac{15}{100} \times 9\,000$	= £1 350	
Less rent paid	= £300	£1 050 +
Taxable income	=	£10 050

Tax charged;

First £5 808, the tax due is sh. 5 808 × 2 =	sh. 11 616
Remaining £4 242, the tax due is sh. 4 242 × 3 =	sh. 12 726 +
	sh. 24 342 ✓
Less personal relief	sh. 12 672 –
	sh. 11 670

$$\text{PAYE} = \text{sh. } \frac{11\,670}{12}$$

$$= \text{sh. } 972.50$$

Example 20

Mr Odhiambo is a senior teacher on a basic monthly salary of Ksh. 16 000. On top of his salary, he gets a house allowance of Ksh. 12 000, a medical allowance of Ksh. 3 060 and a hardship allowance of Ksh. 4 635. He has a life insurance policy for which he pays Ksh. 800 per month and claims insurance relief.

(a) Use the tax table below to calculate his PAYE.

Table 6.2

Income in £ per month	*Rate (%)*
1 – 484	10
485 – 940	15
941 – 1 396	20
1 397 – 1 852	25
Excess over 1 852	30

(b) In addition to the PAYE, the following deductions are made on his pay every month:

(i) WCPS at 2% of basic salary.

(ii) NHIF Ksh. 400

(iii) Co-operative shares and loan recovery Ksh. 4 800.

Calculate his net pay.

Solution

(a) Taxable income = Ksh.(16 000 + 12 000 + 3 060 + 4 635)

= Ksh. 35 695

converting to K£ = $\frac{\text{K£ } 35\ 695}{20}$

= K£ 1 784.75

From table 6.2, tax charged is;

First £ 484 = £ 484 × $\frac{10}{100}$ = £ 48.40

Next £ 456 = £ 456 × $\frac{15}{100}$ = £ 68.40

Next £ 456 = £ 456 × $\frac{20}{100}$ = £ 91.20

Remaining £ 388 = £ 388 × $\frac{25}{100}$ = £ 97.00

Total tax due = £ 305.00

= sh. 6 100

Insurance relief = sh. $\frac{800}{20}$ × 3 = sh. 120

Personal relief = sh. 1 056 +

Total relief = sh. 1 176

Tax payable per month is sh. 6 100

sh. 1 176 –

sh. 4 924

Therefore, PAYE is sh. 4 924.

Note:

For calculation of PAYE, taxable income is rounded down (truncated) to the nearest whole number. For example, K£ 1 784.75 is taken as K£ 1 784.

(b) Total deducations are.

sh. ($\frac{2}{100}$ × 16 000 + 400 + 4 800 + 800 + 4 924 = sh. 11 244

∴ Net pay = sh. (35 695 – 11 244)

= sh. 24 451

If an employees' due tax is less than the relief allocated, then that employee is exempted from PAYE.

Exercise 6.5

Use table 6.1 for the rates of taxation and the given reliefs for this exercise.

1. Calculate the annual tax due for the following:
    (a) Mr. Kemei with taxable income of K£ 8 000 p.a.
    (b) Mrs. Pamba with taxable income of K£ 12 000 p.a.
    (c) Mr. Kigosi with taxable income of K£ 18 460 p.a.
    (d) Ms. Kuria with taxable income of K£ 40 000 p.a.
    (e) Mrs. Mikisi with taxable income of K£ 22 300 p.a.
2. Calculate the PAYE payable by the following:
    (a) Mrs. Ojunga on a monthly salary of sh. 12 400, a house allowance of sh. 8 000 per month and a medical allowance of sh. 2 400 per month.
    (b) Mr. Kitur whose monthly earnings are; a salary of sh. 12 800, house allowance of sh. 6 000, hardship allowance of sh. 3 840 and a medical allowance of sh. 2 100.
    (c) Ms. Okinyo with a salary of sh. 14 800, a house allowance of sh. 12 000 and a medical allowance of 2 400 all per month. She also pays for a life insurance policy at sh. 1 200 per month and claims insurance relief.
    (d) Mr. Mwangi on a monthly salary of sh. 5 000, a medical allowance of sh. 900. He is housed by his employer and pays a nominal rent of sh. 300.
3. The annual income of a businesswoman is £ 8 000. She has a life insurance policy for which she pays a premium of £ 240 p.a. Calculate the income tax she has to pay.
4. The salary of a hotel manager is sh. 9 000 per month. He pays sh. 450 per month towards his life insurance policy and claims insurance relief. Find the income tax he pays in a year.
5. If a man pays £ 60 as income tax per year, find his annual income.
6. For what monthly income is the amount of PAYE chargeable equal to the personal relief?
7. Mr. Kemboi earns sh. 13 750 per month. He is given a monthly house allowance of sh. 5 800 and a car allowance of sh. 2 400 per month.
    (a) How much PAYE does he pay?
    (b) If he had a life insurance policy for which he paid premiums of sh. 900 per month, how much PAYE would he pay?
8. Mr. Kisaka's income is £12 840 p.a. and Mrs. Kisaka's income is £13 140 p.a.

They are housed by the employer and Mr. Kisaka pays a nominal rent of sh. 1 000 per month for the house. Find the total income tax they pay in a year.

9. Mr. Ahmed earns £13 636 p.a. and is housed by the employer. He pays a premium of £230 p.a. towards his life insurance policy and claims insurance relief. Find his PAYE.

10. Mr. Mambo's income in one year came from the following sources:

Rent from his houses	sh. 150 000
Commissions	sh. 48 000
Dividends on shares	sh. 210 000

Calculate how much income tax he should pay.

11. Mr. Musasia is a director of a company and earns a salary of sh. 36 500 per month. He has a life insurance policy for which he pays sh. 2 000 per month. The company provides him with a free house, water, electricity and a 2 000 cc limousine. His other taxable benefits are as follows:

Car – £250 per month
Water – £20 per month
Electricity – £50 per month.

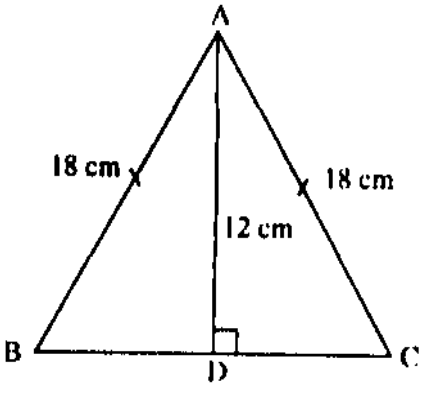
(a) How much PAYE does he pay?

(b) If apart from the insurance and PAYE in (a) other deductions on his pay add up to sh. 11 400 per month, calculate his net monthly pay.

Mixed Exercise 1

1. Form the quadratic equations whose roots are:
   (a) 5 and $-\frac{1}{3}$.
   (b) -2 and -3.
   (c) 7 and -11.
2. Find simple interest on each of the following:
   (a) sh. 500 at 10% p.a. for 3 years.
   (b) sh. 1 400 at 5% p.a. for 2 years.
   (c) sh. 20 000 at 8% p.a. for $5\frac{1}{2}$ years.
   (d) sh. 25 000 at $7\frac{1}{2}$% p.a. for 5 years.

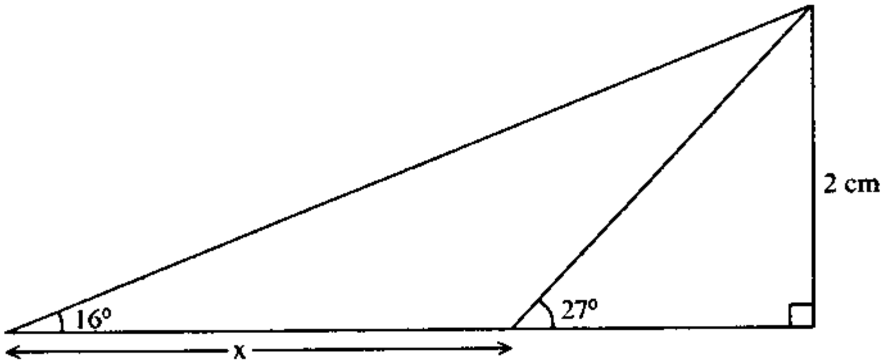
3. Find the minimum possible perimeter of a regular hexagon whose side measures 12.6 cm to one decimal place.
4. Round off the following to four significant figures:
   (a) 21.736 (b) 0.0070923 (c) 430 947
   (d) 1.2047 (e) 0.47291 (f) $\frac{5}{7}$
5. Find the amount after 5 years when sh. 30 000 is invested at:
   (a) 11% p.a. simple interest.
   (b) 11% p.a. compound interest.
6. The figure below shows an isosceles ΔABC in which $AB = AC = 18$ cm. If the height AD of the triangle is 12 cm, find the sizes of the angles of the triangle.



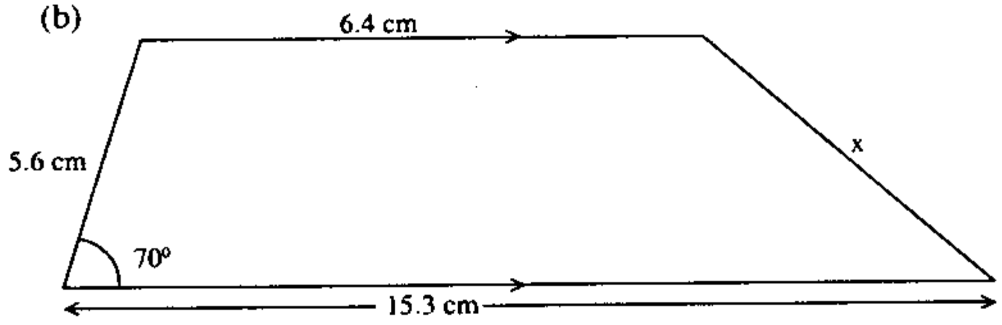
7. Show that $(\sqrt{p} + \sqrt{q})^2 = p + q + 2\sqrt{pq}$.
8. Truncate the following to three decimal places:
   (a) $\frac{2}{3}$ (b) 2.14732 (c) 523.9746 (d) 17.3489 (e) 0.0006374
9. The roots of the quadratic equation $x^2 + px - q = 0$ are 1 and -5. Find the values of p and q.

10. A bicycle can be bought for sh. 4 500 cash or by 12 equal monthly instalments of sh. 400. Calculate the carrying charge.
11. If θ lies between 0^0 and 180^0 and $\sin \theta = 0.5$, find the two possible values of θ.
12. A triangle ABC is right-angled at B. AB = 5 cm and AC = 13 cm. Find the area of the triangle using each of the following formulae:
    (a) $A = \frac{1}{2}bh$
    (b) $A = \frac{1}{2}ab \sin C$
    (c) $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are sides of the triangle, h the height and s the semi-perimeter.
13. The difference between two numbers is 5. If their product is –6, find the numbers.
14. Find x in each of the following figures:

(a)



(b)



15. Evaluate sin $\frac{\sin 60^0 \ \cos 30^0}{\tan 30^0 \ \sin 45^0}$, leaving your answer in surd form.
16. Find y if $\log_2 y - 2 = \log_2 92$
17. Factorise the following quadratic expressions:
    (a) $8x^2 + 6x - 9$ (b) $3x^2 - \frac{7}{4}x + \frac{1}{8}$ (c) $\frac{1}{6}x^2 - \frac{1}{6}x - 1$ (d) $16 - 4x^2$
18. Use tables to solve the following correct to the nearest one tenth of a degree.

(a) $8 \sin \theta = 3$ (b) $16 \cos \theta - 12 = 0$ (c) $2 + 3 \tan \theta = 0$

(d) $8 \sin \theta - 5 = 0$

19. Fill in the blanks the missing terms in the following to make them perfect squares:

(a) $x^2 \_\_ + 4$ (b) $\_\_ - 12x + x^2$ (c) $9x^2 \_\_ + \frac{1}{4}$ (d) $x^2 - \frac{2}{3}x \_\_$

20. A fridge costs sh. 14 000. It may be bought at hire purchase by paying a deposit of 3 500 and the remainder, which has an interest charge of 18%, added, in 12 equal monthly instalments. Calculate:

(a) the interest.

(b) the monthly instalments.

21. Given that $\tan x = \frac{5}{12}$, find the value of:

(a) $\cos x$ (b) $\sin x$ (c) $\cos (90 - x)$ (d) $\sin (180 - x)$

22. The value of a car is sh. 54 000 to the nearest sh. 1 000. Find the lower and upper limits of the value of the car.

23. Solve the following equations:

(a) $x^2 - 8x - 30 = 0$ (b) $x^2 + x - 1 = 0$ (c) $4x^2 + 6x + 1 = 0$

(d) $16x^3 + 8x^2 + x = 0$ (e) $x^2 - 5x + 6 = 0$ (f) $2x^2 - 9x - 5 = 0$

(g) $\frac{x-1}{1} = \frac{1}{2x-3}$

24. The dimensions of a cuboid are stated as 4.8 cm by 3.6 cm by 2.5 cm to 1 d.p. within what limits does its:

(a) volume, and

(b) surface area lie?

25. Solve for x;

(a) $\frac{2}{x-5} + \frac{3}{4x-1} = \frac{1}{5}$ (b) $\frac{x}{x+5} + \frac{x+5}{x} = 10$

26. Four measurements p, q, r and s measured to the nearest 0.1 cm are stated as p = 1.8, q = 2.4, r = 9.2 and s = 5.0. Find the relative errors in each of the following:

(a) $p + q$ (b) $r - s$ (c) $s - p + q$ (d) rp

(e) pqr (f) $r(p + s)$ (g) $rq - ps$

27. Write in the form $a\sqrt{b}$, where a and b are integers:

$$\frac{25^{\frac{1}{2}} \times 4^{\frac{1}{2}} \times 32^{\frac{1}{2}}}{8^{\frac{1}{3}}}$$

28. Given that log y = 3.143 and log x = 2.421, evaluate:

(a) $4 \log y^{\frac{1}{2}} + \log \sqrt[3]{a}$ (b) $\log x^4 - \frac{1}{4} \log y^3$

29. A man earns K£ 12 928 p.a. Using the current income rates, calculate the P.A.Y.E. per month. (You may use the rates given in this book).
30. A student recorded the length of a paper as 20.67 cm. If the actual measurement was $20\frac{2}{3}$ cm, calculate the relative error to two significant figures.
31. If $\cos\theta = \frac{15}{17}$, and $270^0 \le \theta \le 360^0$, find $\sin\theta$ and $\tan\theta$.
32. The length of a rectangle is twice its width plus one. If the area of the rectangle is 21 cm^2, find the perimeter of the rectangle.
33. Solve the triangle XYZ in which $y = 9$ cm, $z = 6.5$ cm and $\angle YXZ = 72^0$.
34. Find the smallest and the greatest difference between:
    (a) 28.0 cm and 16.4 cm.
    (b) 0.05 km and 0.082 km.
    (c) 4.36 kg and 2.37 kg.
35. Given that $\log y^3 = \log \sqrt[3]{81}$, find y.
36. A housewife ordered 1.75 m material for a table cloth. If the actual measurement was $1\frac{2}{3}$ m, calculate the percentage error.
37. Solve the triangle PQR in which $\angle PQR = 32^0$, $q = 7$ cm and $r = 12$ cm (2 possible triangles).
38. The angles of a triangle are in the ratio 3 : 4 : 2. If the shortest side is 5 cm, calculate the length of the longest side.
39. Simplify the following without using tables:

    (a) $\dfrac{\log 27 - \log 9}{\log 3}$ (b) $\dfrac{\frac{1}{2}\log 625 + \frac{1}{3}\log 125}{\log 100 - \frac{1}{2}\log 16}$

40. Draw the graph of $\sin\theta$ for $0^0 \le \theta \le 360^0$ and use it to solve:

    (a) $4\sin\theta = 3$ (b) $\sin\theta = \frac{1}{3}$ (c) $16\sin\theta + 4 = 0$

41. The present ages of two children are 2 and 5 years respectively. After how long will the sum of the squares of their ages be 45?
42. Simplify:

    (a) $3\sqrt{15} + 5\sqrt{735} - \sqrt{375}$ (b) $\dfrac{3}{3+\sqrt{2}} - \dfrac{1}{3-\sqrt{2}}$

    (c) $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$ (d) $\dfrac{2}{4-\sqrt{5}}$ (e) $\dfrac{3}{4-\sqrt{2}} + \dfrac{\sqrt{2}}{4-\sqrt{2}}$

43. Solve the following equation:

    (a) $2\log x + \frac{1}{2}\log 81 = \log 6x$

    (b) $\log(x-2) + \log(x+6) - \log 5 = 0$

44. The sides of a rectangle are 31.4 cm by 28.3 cm. Within what limits does its perimeter [illegible]

45. A man wishes to save sh. 200 000 in 4 years time. Find the sum of money he has to deposit now at 12% p.a. interest, compounded semi-annually to realise his goal.

46. A financial institution gives a building loan of sh. 3 600 000 to Mr. Mukolwe to be paid in 20 years. If the compound interest rate is 12% p.a., calculate the total amount paid to the financial institution at the end of the 20 years and the amount he pays each month.

47. Find by graphical method the solutions to each of the following:
(a) $2x^2 + 3x + 1 = 0$ (b) $x^2 + 3x + 2 = 0$ (c) $3x^2 - 4x - 7 = 0$

48. Use graphical method to solve the following simultaneous equations:
(a) $y = x^2 + 2x + 1$ (b) $y = 3x^2 + 2x - 6$
$y = 2x + 1$ $y = x + 2$
(c) $y = x^2 + 2x - 2$ (d) $y = x^3 + 2x^2 + x$
$y = 2 - 2x - x^2$ $y = 3x$

49. Given that $\sqrt{2} = 1.414$ and $\sqrt{3} = 1.732$, evaluate correct to 4 s.f.:
(a) $\frac{1}{\sqrt{3} - \sqrt{2}}$ (b) $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}$

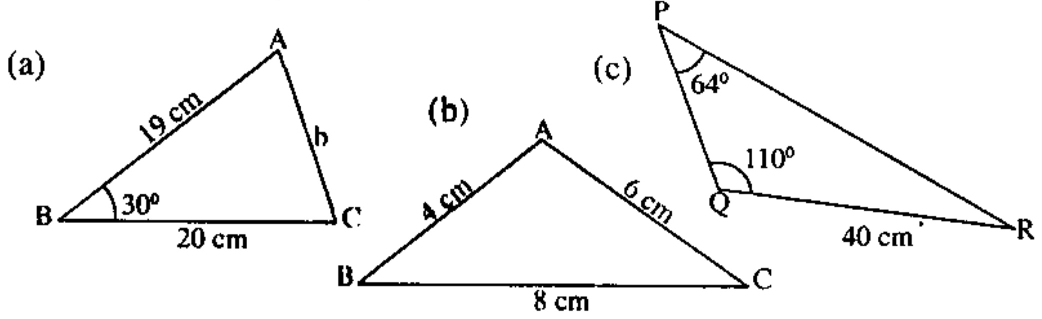
50. Find the minimum and the maximum possible sums of the following:
(a) 8.62 cm and 5.87 cm.
(b) 250 m, 182.4 m and 218.5 m.
(c) 35.2 litres and 45.9 litres.

51. A boat sails 16 km N 35° E and then 30 km N 21° W. How far is the boat east and north of the starting point?

52. A parallelogram has a base of 12 cm and height of 8 cm.
(a) Find the limits within which its area lies.
(b) Find the relative error in the area.

53. Express in surd form:
(a) $\frac{1}{2 + \sin 45^0}$ (b) $\frac{3}{1 - \cos 60^0}$

54. Solve the following triangles:



CIRCLES: CHORDS AND TANGENTS

7.1: Length of an Arc

Figure 7.1 below shows a circle, centre O and radius r units.

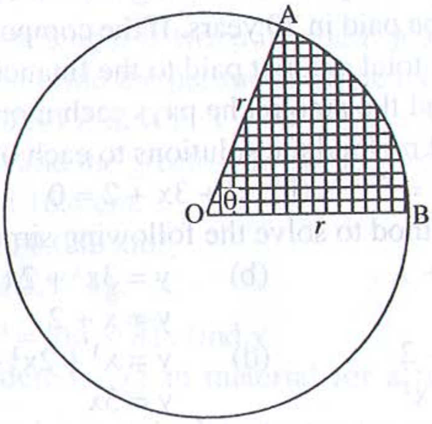


Fig. 7.1

The length of the minor arc AB subtending an angle θ at the centre of the circle is $\frac{\theta}{360} \times 2\pi r$.

Example 1

Find the length of the arc subtending an angle 250^0 at the centre of circle of radius 14 cm. (Take $\pi = \frac{22}{7}$)

Solution

Length of an arc = $\frac{\theta}{360} \times 2\pi r$

$\theta = 250^0$ and r = 14 cm

$\therefore$ Length of the arc = $\frac{250}{360} \times 2 \times \frac{22}{7} \times 14$

$= 61.11$ cm

Example 2

The length of an arc of a circle is 11.0 cm. Find the radius of this circle if the arc subtends 90^0 at the centre of the circle. (Take $\pi = \frac{22}{7}$)

Solution

Arc length = $\frac{\theta}{360} \times 2\pi r$

But $\theta = 90^0$

Therefore, $11.0 = \frac{90}{360} \times 2 \times \frac{22}{7} \times r$

$r = 7.0$ cm

Example 3

Find the angle subtended at the centre of a circle by an arc of length 20 cm if the circumference of the circle is 60 cm.

Solution

$\frac{\theta}{360} \times 2\pi r = 20$

But $2\pi r = 60$ cm

Therefore, $\frac{\theta}{360} \times 60 = 20$

$\theta = 20 \times \frac{360}{60}$

$\theta = 120^0$

Exercise 7.1

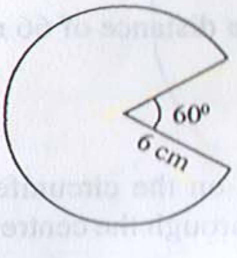
1. Copy and complete the following table:

Table 7.1

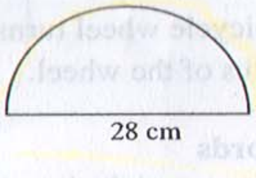
Angle subtended by arc at centre, θ	*Circumference (c) in cm*	*Arc length in cm* $\left(\frac{\theta}{360} \times c\right)$
30^0	48	–
250^0	25	–
45^0	–	7
90^0	–	10
–	60	20
–	30	15

2. Find the perimeters of each of the following figures:

(a)



(b)



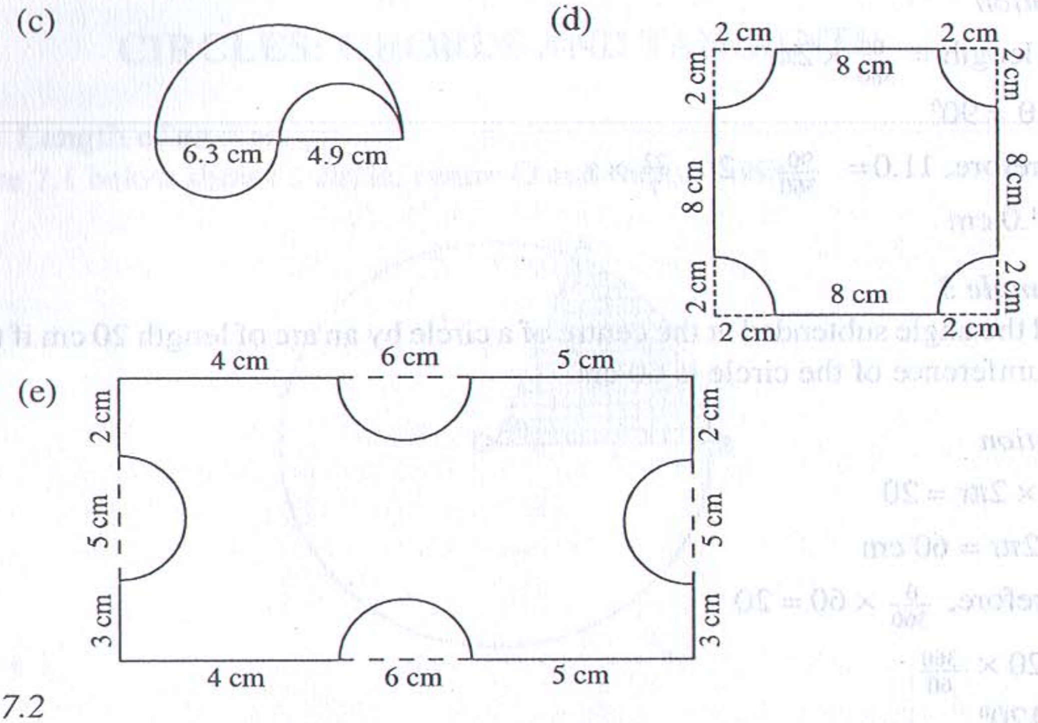


Fig. 7.2

3. Find the perimeter of a semi-circular protractor whose radius is 14 cm.
4. Find the radius of a bicycle wheel whose circumference is 198 cm.
5. An arc PQ of a circle of radius 15 cm subtends an angle 160^0 at the centre of the circle. Find the length of the arc PQ.
6. A wheel of diameter 14 cm is rotating at 2 500 revolutions per minute. Express the speed of a point on the rim in centimetres per second.
7. Two circular wires of diameters 9 cm and 12 cm are cut and joined to make one large circle. Find the radius of this circle.
8. The perimeter of a semi-circular protractor is 14.28 cm. Find its radius.
9. An arc of a circle is 6 cm. It subtends an angle of 72^0 at the centre of the circle. Find the radius of the circle.
10. The wiper of a bus is 40 cm long. It sweeps out through an angle of 120^0 on a flat windscreen. Calculate the distance moved by the tip of the wiper in one 'sweep'.
11. A bicycle wheel turns 15 times in covering a distance of 66 m. Find the radius of the wheel.

7.2: Chords

A chord is a straight line joining any two points on the circumference of a circle. The diameter is a special chord that passes through the centre of a circle.

Perpendicular Bisector of a Chord

A perpendicular drawn from the centre of a circle to a chord bisects the chord. Consider figure 7.3 below. It represents a circle in which PQ is a chord.

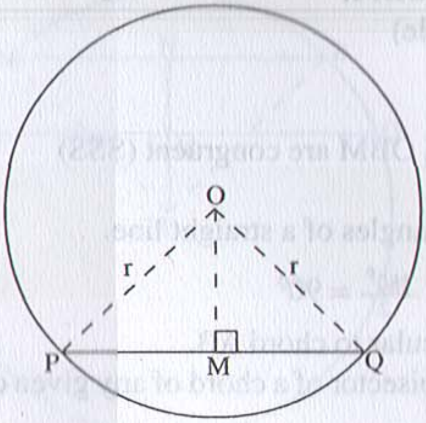


Fig. 7.3

In the figure, OM is perpendicular to PQ.
In Δs POM and QOM;
OP = OQ (radii of circle) and OM is common.
∠OMQ = ∠OMP = 90⁰ (construction)
Therefore, Δs POM and QOM are congruent (RHS).
Hence, PM = QM.

In general a perpendicular drawn from the centre of a circle to a chord bisects the chord. Conversely, a line that bisects a chord passes through the centre of a circle.

By considering the cosine of angles OPM and OQM, use trigonometry to prove the above.

A straight line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

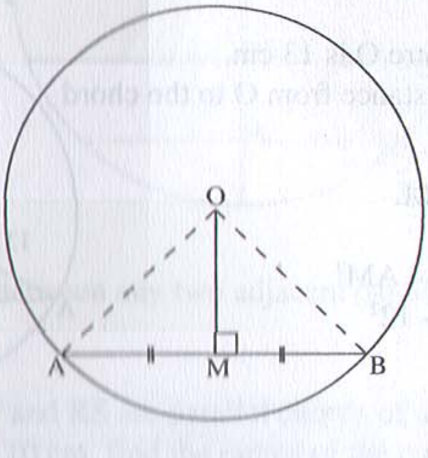


Fig. 7.4

In figure 7.4, M is the midpoint of chord AB. OM is the straight line through M passing through the centre O of the circle.

In triangles OAM and OBM;

OA = OB (radii of circle)

AM = MB (given)

OM is common

$\therefore$ Triangles OAM and OBM are congruent (SSS)

$\angle OMA = \angle OMB$

But these are adjacent angles of a straight line.

$\therefore \angle OMA = \angle OMB = \frac{180^0}{2} = 90^0$

Thus, OM is perpendicular to chord AB.

Thus, a perpendicular bisector of a chord of any given circle passes through the centre of the circle.

Example 4

In figure 7.5, XY is a chord of a circle centre O and radius 10 cm. If the perpendicular distance ON is 6 cm, calculate the length of the chord.

Solution

From the figure;

$ON^2 + NY^2 = OY^2$ (Pythagoras' theorem)

$6^2 + NY^2 = 10^2$

$NY^2 = 10^2 - 6^2$

$= 64$

$\therefore NY = 8$ cm

But $XY = 2NY$

$\therefore XY = 16$ cm

The chord is 16 cm long.

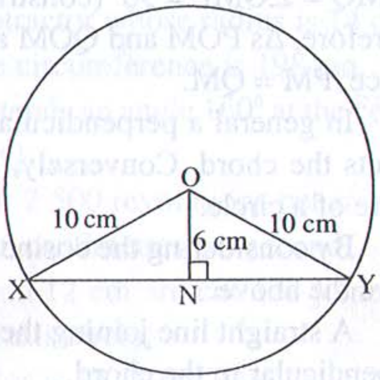


Fig. 7.5

Example 5

The radius of a circle centre O is 13 cm.

Find the perpendicular distance from O to the chord

Solution

OM bisects chord AB at M.

Therefore, AM = 12 cm

In ΔAOM, $OM^2 = AO^2 - AM^2$

$= 13^2 - 12^2$

$= 25$

Therefore, $OM = \sqrt{25}$

$= 5$ cm

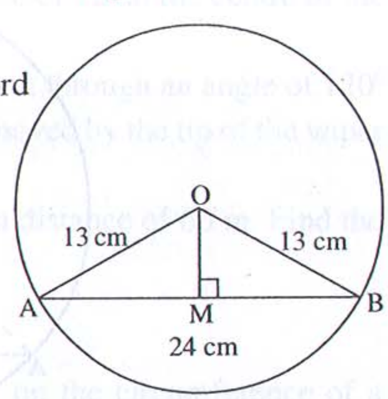


Fig. 7.6

Parallel Chords

Consider figure 7.7 below. OM is perpendicular to PQ and PQ is parallel to AB.

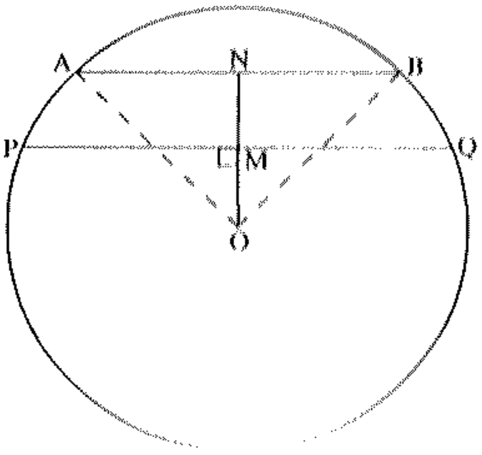


Fig. 7.7

If OM is produced to meet AB at N, then ∠OMQ = ∠ONB = 90^0 (corresponding angles). OA = OB (radii of circle), and ON is common.
Therefore, Δs ONB and ONA are congruent (RHS). Hence, AN = NB and ∠ ONA = ∠ONB = 90^0. Thus, ON is a perpendicular bisector to chords AB and PQ.

In general, any chord passing through the midpoints of all parallel chords of a circle is a diameter, see figure 7.8.

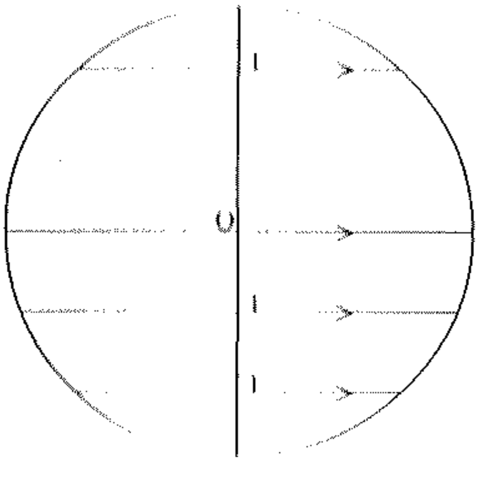


Fig. 7.8

Note also that the arcs between any two adjacent parallel chords are equal.

Example 6

In figure 7.9 below, PQ and RS are parallel chords of a circle and 2 cm apart. If PQ = 8 cm and RS = 10 cm, find the radius of the circle.

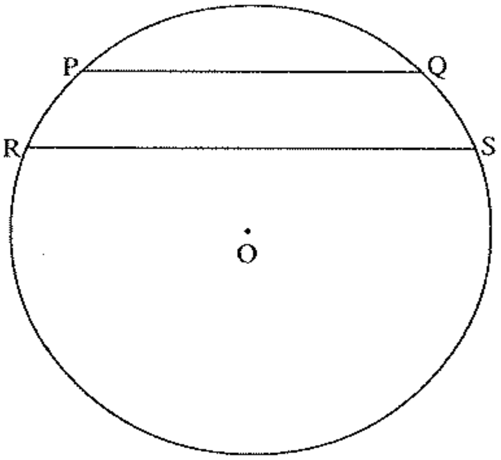


Fig. 7.9

Solution

Draw the perpendicular bisector of the chords to cut them at K and L as in figure 7.10.

Join OP and OR

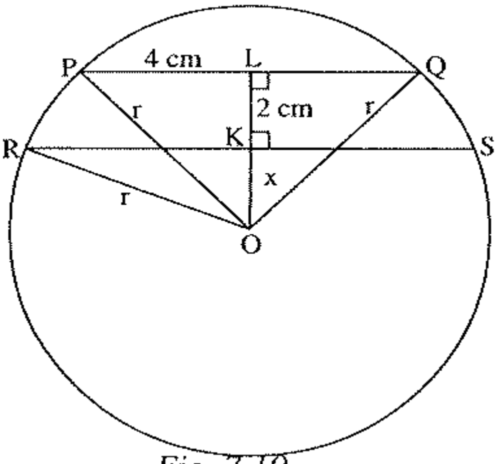


Fig. 7.10

In ΔOPL,

PL = 4 cm and KR = 5 cm

Let OK = x cm

$\therefore (x + 2)^2 + 4^2 = r^2$

In ΔORK;

$x^2 + 5^2 = r^2$

$\therefore (x + 2)^2 + 4^2 = x^2 + 5^2$

$x^2 + 4x + 20 = x^2 + 5^2$

$4x + 20 = 25$

$4x = 5$

$x = 1\frac{1}{4}$

Using the equation $r^2 = x^2 + 5^2$;

$$r^2 = (\tfrac{5}{4})^2 + 5^2$$

$$= \tfrac{25}{16} + 25$$

$$= \tfrac{425}{16}$$

Therefore, $r = \sqrt{\frac{425}{16}}$

$= 5.154$ cm

Example 7

Two parallel chords of a circle are each 8 cm long. If the radius of the circle is 5 cm, what is the perpendicular distance between the chords?

Solution

Consider ΔLOB in figure 7.11 below.

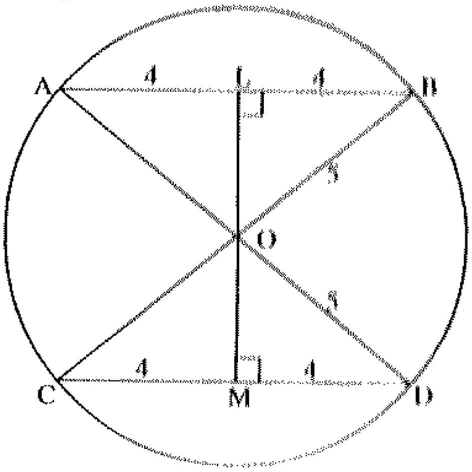


Fig. 7.11

LB = 4 cm (LO is bisector of AB).
OB = 5 cm
$LB^2 + LO^2 = OB^2$
Therefore, $4^2 + LO^2 = 5^2$
$16 + LO^2 = 25$
$LO^2 = 9$
LO = 3 cm
Similarly, OM = 3 cm
∴ LM = LO + OM = 6 cm
The chords are 6 cm apart.

Equal Chords

Angles subtended at the centre of a circle by equal chords are equal.

(i) Figure 7.12 is a circle centre O. AB and CD are equal chords.

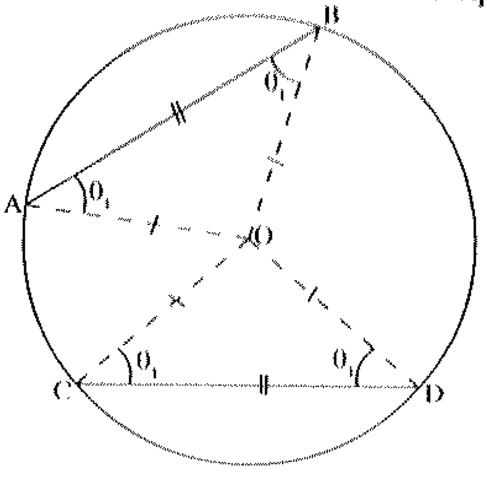


Fig. 7.12

In Δs AOB and COD, OA = OB (radii of the same circle).
$\angle OAB = \angle OBA = \theta_1$ (base $\angle$s of isosceles triangle)
Similarly, OC = OD and $\angle OCD = \angle ODC = \theta_1$
$\Delta AOB \equiv \Delta COD$ (ASA)
$\therefore \angle AOB = \angle COD$ (third angles equal).

Hence, the angles subtended at the centre by equal chords are equal.
(ii) Figure 7.13 shows a circle centre O. Chords PQ and RS are equal.

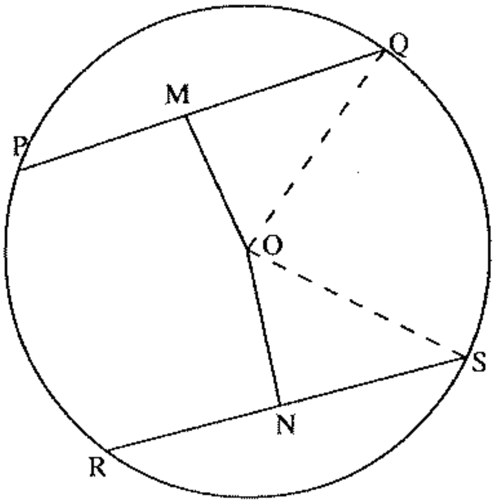


Fig. 7.13

Using Δs OMQ and ONS;
OS = OQ (radii of same circle), and NS = MQ (half length of equal chords).
$\angle OMQ = \angle ONS$ (ON and OM are $\perp$ bisectors of RS and PQ respectively).
$\Delta OMQ = \Delta ONS$ (RHS).
$\therefore$ OM = ON.

Hence, if chords of a circle are equal, they are equidistant from the centre of the circle. Conversely, if two chords of a circle are equidistant from the centre, then they are equal in length.

Intersecting Chords

(i) Figure 7.14 shows a circle with the chords XY and LM intersecting internally at K.

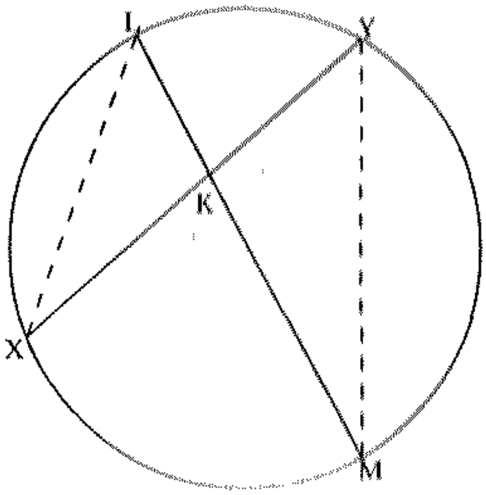


Fig. 7.14

From the Δs LKX and YKM;
∠XLK = ∠MYK (angles subtended by the same arc XM)
∠KXL = ∠KMY (angles subtended by the same arc LY)
∠LKX = ∠YKM (vertically opposite angles)
∴ ΔLKX is similar to ΔYKM (equiangular triangles)

So, $\frac{YK}{LK} = \frac{KM}{KX}$
∴ YK × KX = KM × LK

Example 8
AB and XY are two chords that intersect in a circle at R. Given that AR = 4 cm, XR = 5 cm and RY = 3 cm, find AB.

Solution

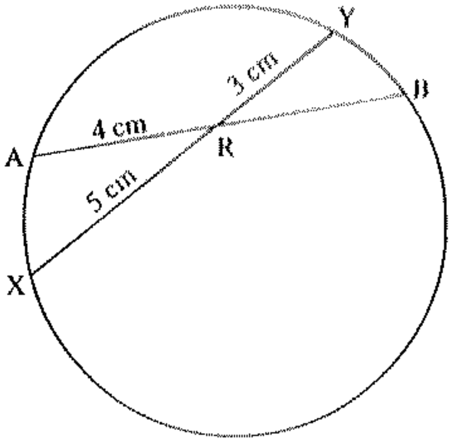


Fig. 7.15

Let RB = x cm
$4 \times x = 5 \times 3$
$4x = 15$
$x = 3.75$ cm

Since AB = AR + RB
$\therefore$ AB = 4 + 3.75
= 7.75 cm

(ii) Figure 7.16 shows a circle. AB and PQ are chords intersecting externally at a point C.

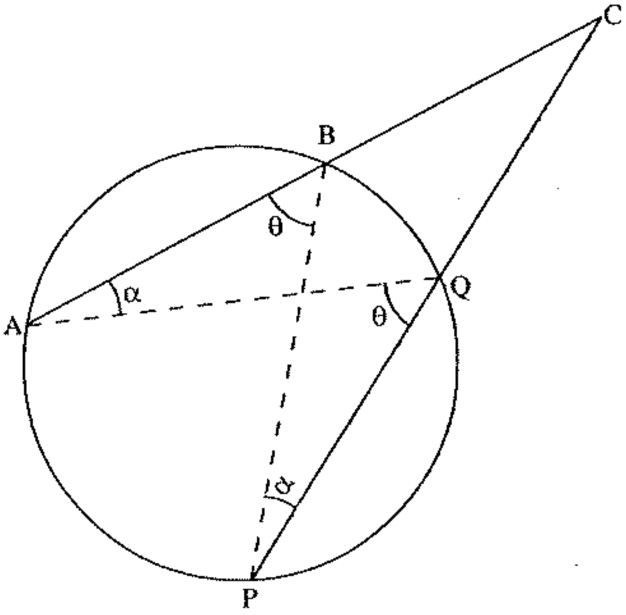


Fig. 7.16

Now, consider ΔQCA and BCP:
$\angle BAQ = \angle QPB = \alpha$ (angles subtended by the same arc BQ)
$\angle PQA = \angle ABP = \theta$ (angles subtended by the same arc AP)
$\therefore \angle AQC = \angle PBC = 180^0 - \theta$
$\angle ACQ = \angle PCB$ (common angle to both triangles)
So, ΔACQ is similar to ΔPCB (equiangular triangles)

$$\frac{QC}{BC} = \frac{CA}{CP}$$

$\therefore QC \times CP = CA \times BC$

Note:
A chord that is produced outside a circle is called a secant.

Example 9

Find the value of x in figure 7.17 below. The chords AB and CD intersect externally at point O

Solution

$OA \times OB = OC \times OD$

$(9 + x)\,x = (5 + 4)4$

$x^2 + 9x = 36$

$x^2 + 9x - 36 = 0$

$(x + 12)(x - 3) = 0$

Therefore, $x = -12$ or $x = 3$

x can only be 3 cm.

Thus, x = 3 cm.

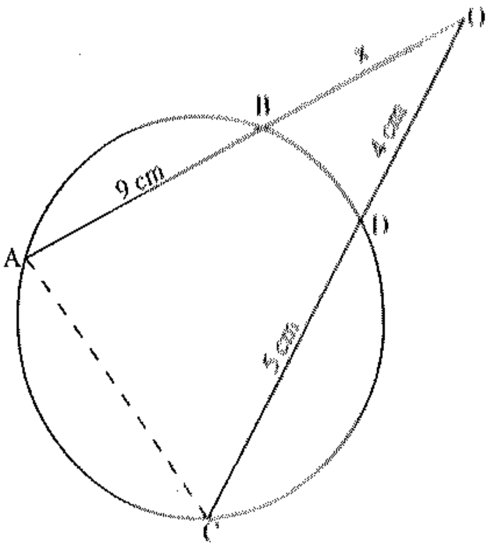


Fig. 7.17

Exercise 7.2

1. A chord 12 cm long is on a circle of radius 10 cm. Find the distance of the chord from the centre of the circle.
2. Find the length of a chord of a circle radius 17 cm whose distance from the centre of the circle is 8 cm.
3. Find the length of a chord of a circle radius $\sqrt{x^2 + y^2}$ cm whose distance from the centre of the circle is y cm.
4. XY and RS are parallel chords on opposite sides of the centre of a circle of radius 13 cm. If XY = 24 cm and RS = 20 cm, find the distance between the chords.
5. Figure 7.18 shows a circle centre O. OS, the mediator of PQ, cuts PQ at R. If PQ = 30 cm and RS = 9 cm, calculate the radius of the circle.

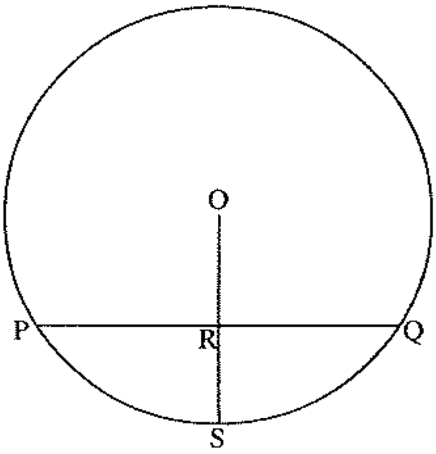


Fig. 7.18

6. Figure 7.19 shows a circle centre O radius 40 cm. The chord XY is 26 cm. Calculate the area of ΔOXY.

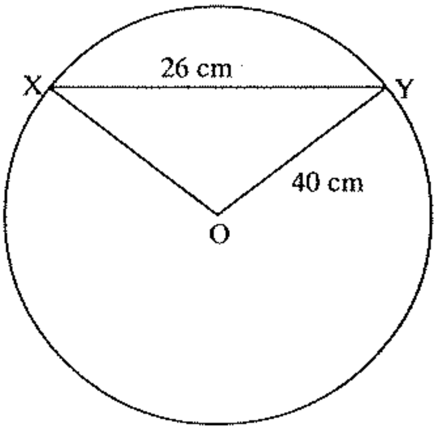


Fig. 7.19

7. Two parallel chords of length 4.4 cm and 9 cm are in a circle radius 7.5 cm.
   (a) Calculate the distance between the two chords if:
      (i) they are on the same side of the centre.
      (ii) they are on opposite sides of the centre.
   (b) Show that the arcs between the parallel chords on the same side of the centre of the circle are equal.
8. Figure 7.20 shows an isosceles triangle ABC in a circle with the vertices A, B and C on the circumference. Calculate the radius of the circle.

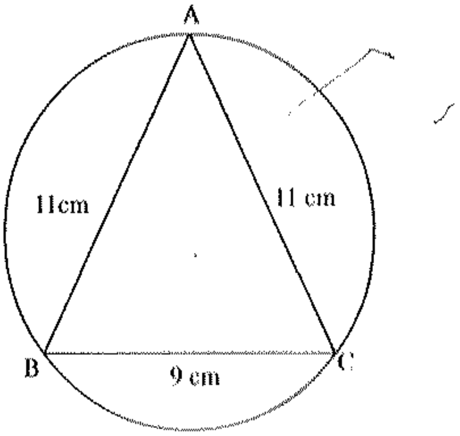


Fig. 7.20

9. Figure 7.21 shows a circle ABCD in which the chords AB and CD are equal. Show that:
   (a) AD = BC.
   (b) AC is parallel to BD.

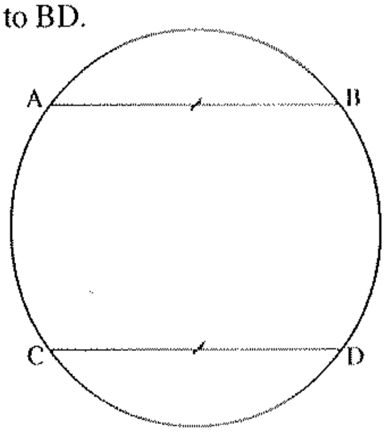


Fig. 7.21

10. Figure 7.22 shows a circle centre O radius 3.4 cm. The chord XY is 6 cm long. LM is a perpendicular bisector of the chord. Calculate the lengths XL and XM.

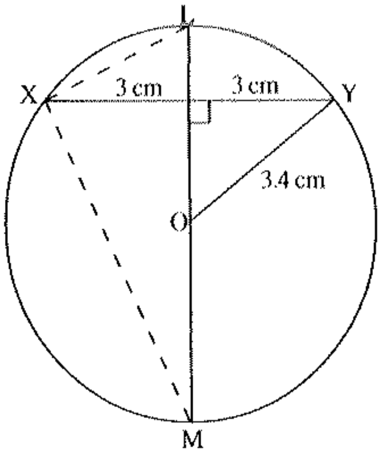


Fig. 7.22

11. Figure 7.23 is a circle centre O and diameter EOC. AB = ED, ∠BOC = 22° and ∠OED = 64°:

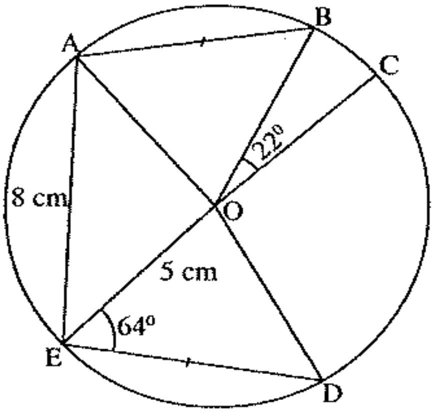


Fig. 7.23

(a) Calculate the value of:
    (i) ∠BOA
    (ii) ∠DOC
    (iii) ∠AOE
  (b) If OE = 5 cm and AE = 8 cm, find the height of triangle OAE with AE as the base.

12. Two chords PQ and RS of the same circle are 11 cm and 13 cm long respectively. If they meet at T in the circle and TR is 3 cm, find PT.

13. Chords AB and CD in figure 7.24 intersect externally at Q. If AB = 5 cm, BQ = 6 cm and DQ = 4 cm, calculate the length of chord CD.

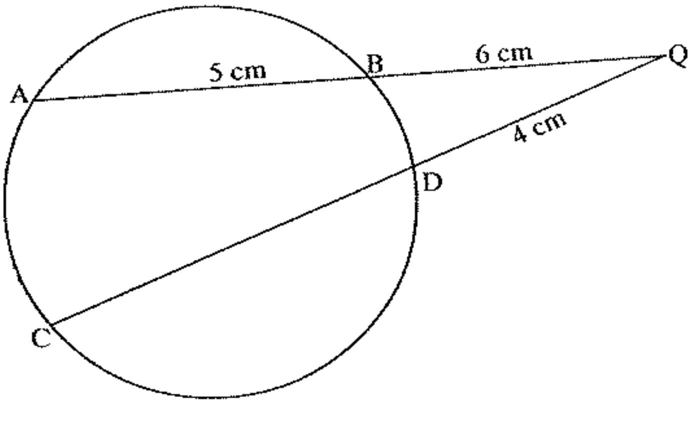


Fig. 7.24

14. Figure 7.24 shows triangle PQR with its vertices on the circumference of a circle. If PQ = PR = 5 cm, QR = 6 cm and PS is the diameter of the circle, calculate:
  (a) TS.
  (b) the radius of the circle.

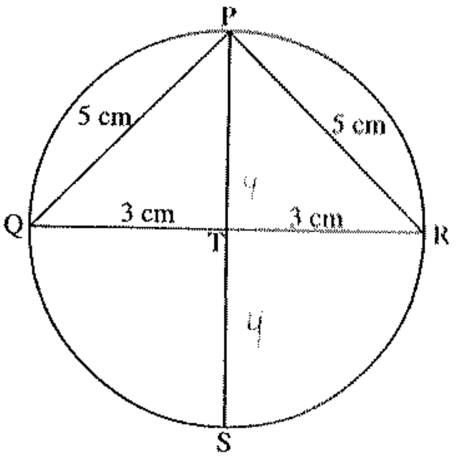


Fig. 7.25

15. Figure 7.26 shows a hemispherical bowl containing water of depth MC. If CM : MO = 2 : 3 and AB = 8 cm, find the radius of the bowl.

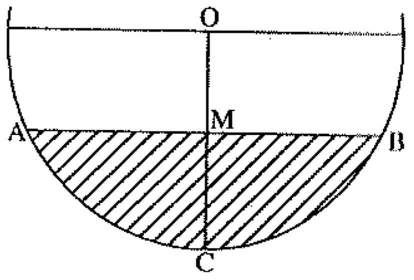


Fig. 7.26

7.3: Tangent to a Circle

In figure 7.27, the line PQ meets and cuts the circle at the points A and B. Such a line is called a secant.

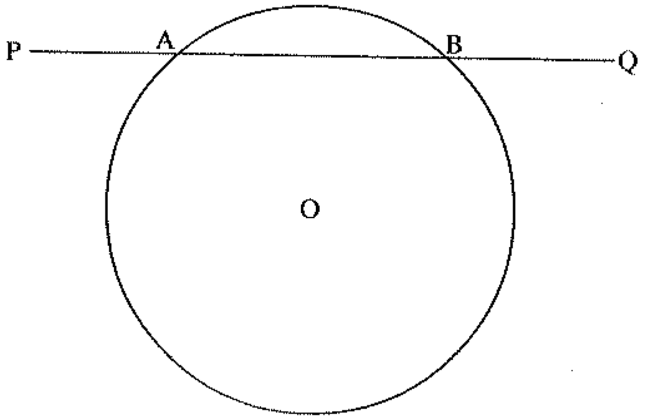


Fig. 7.27

In figure 7.28, the line PQ meets and touches the circle at the point X. Such a line is called a tangent to the circle at point X.

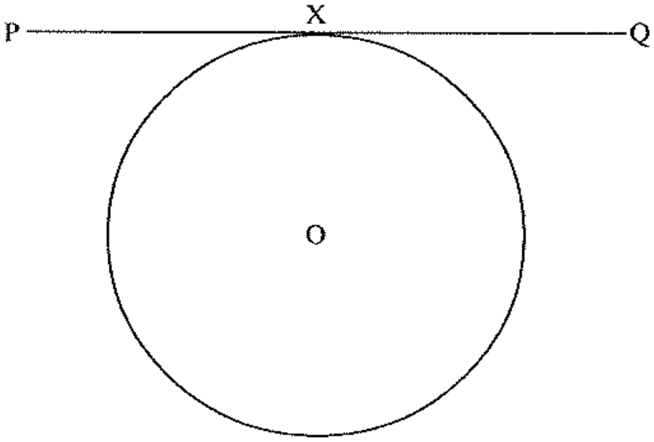


Fig. 7.28

Constructing a Tangent to a Circle

To construct a tangent to a circle, the following method is used (see figure 7.29):

(i) Draw a circle of any radius and centre O.
(ii) Join O to any point X on the circumference.
(iii) Produce OX to a point Y outside the circle.
(iv) Construct a perpendicular line AB through point X.

The line AB is a tangent to the circle at X as shown.

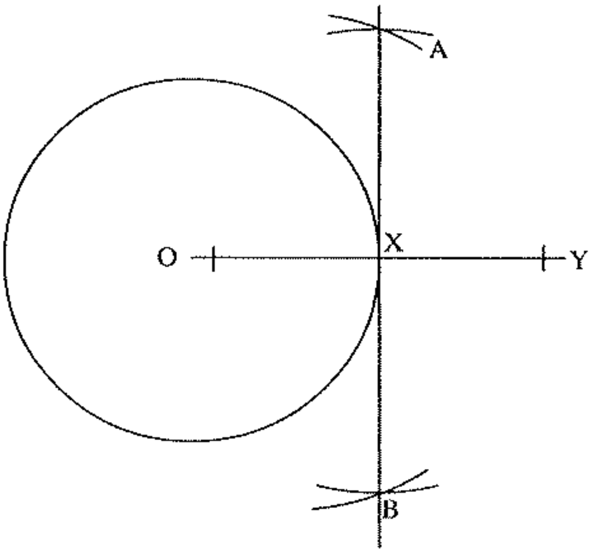


Fig. 7.29

From figure 7.29 it can be deduced that

(i) the radius and the tangent are perpendicular at the point of contact.
(ii) through any one point on a circle, only one tangent can be drawn.

(c) a perpendicular to a tangent at the point of contact passes through the centre of the circle.

Example 10

In figure 7.30, AB is the tangent to the circle centre O. If ∠BMN = 78⁰, find x.

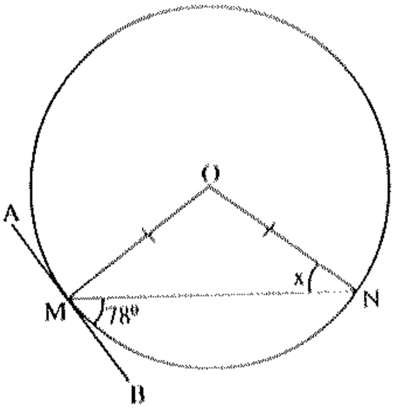


Fig. 7.30

Solution

∠OMB = 90⁰ (radius perpendicular to the tangent).

∠OMN = ∠ONM = x (base angles of an isosceles triangle).

But ∠OMN = (90⁰ – 78⁰) (complementary ∠s.).

= 12⁰

Therefore, x = 12⁰

Example 11

In figure 7.31, calculate the length of PO if PR is a tangent, given that PR = 15 cm and OR = 17 cm.

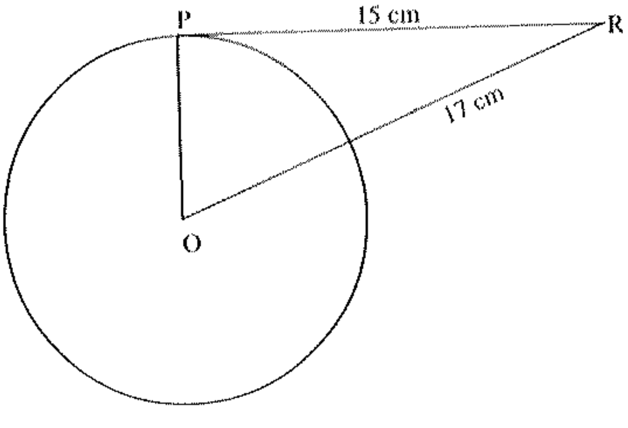


Fig. 7.31

Solution

OP ⊥ PR (radius ⊥ tangent).

$\therefore OP^2 = OR^2 - PR^2$

$= 17^2 - 15^2$

$= 64$

Therefore, OP = 8 cm.

Given a circle centre O and an external point X, construct two tangents from X to the circle.

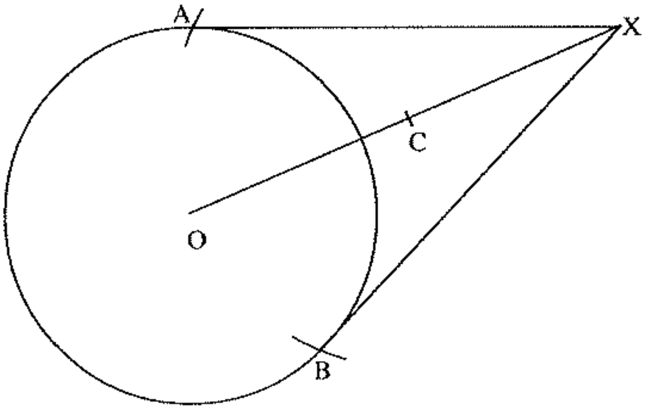


Fig. 7.32

(i) Join X to O and bisect XO to get point C.

(ii) Using C as the centre and radius CO, draw two arcs to cut the circle at points A and B.

(iii) Join A to X and B to X.

AX and BX are the required tangents to the circle from the external point X.

Properties of tangents to a circle from an external point

Example 12

In figure 7.33, LT and MT are tangents to a circle centre O. They meet at an external point T.

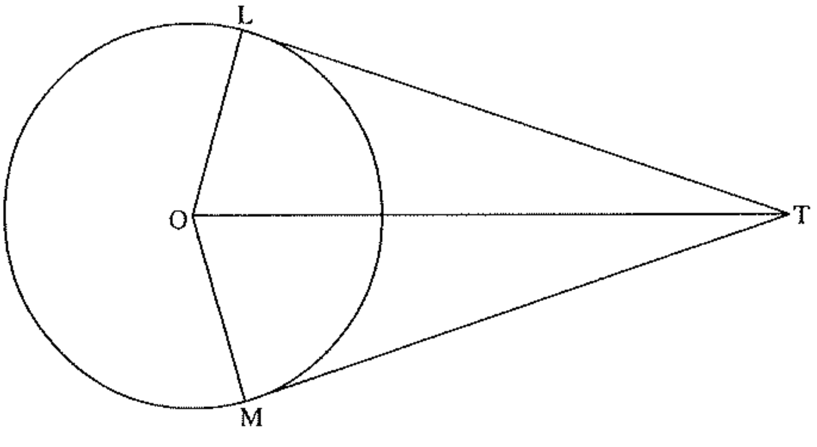


Fig. 7.33

Show that:
(a) LT = MT
(b) ∠TOM = ∠LOT
(c) OT bisects angle LTM.

Solution

Considering triangles TLO and TMO;
∠OLT = ∠OMT = 90^0 (radius ⊥ tangent)
OT is common to both triangles,
OL = OM (radii).
Therefore, triangles TLO and TMO are congruent (RHS)
Hence:
(a) LT = MT
(b) ∠TOM = ∠LOT
(c) ∠OTL = ∠OTM. Therefore, OT bisects ∠LTM.

Note:

If two tangents are drawn to a circle from an external point:
(i) they are equal.
(ii) they subtend equal angles at the centre.
(iii) the line joining the centre of the circle to the external point bisects the angle between the tangents.

Example 13

Find the size of the angles marked x, y and z in figure 7.34 given that ∠ROM = 60^0.

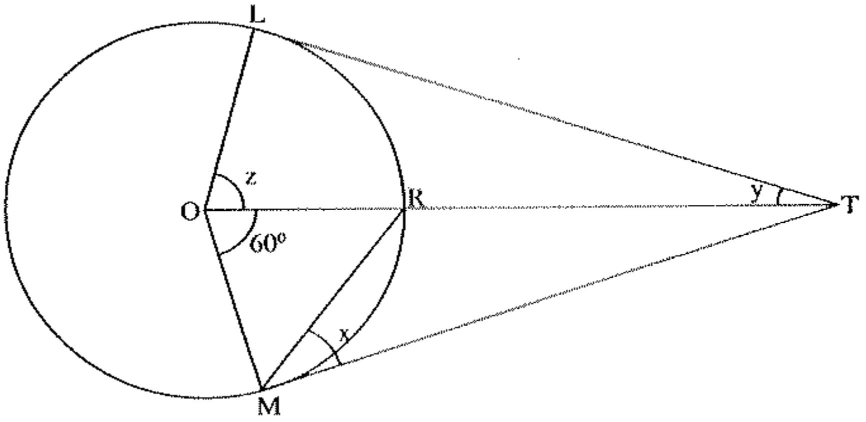


Fig. 7.34

Solution

z = 60^0 (∠s subtended at the centre by equal tangents)
∠OLT = 90^0 (radius ⊥ to tangent LT at L)

$y = 90^0 - 60^0$ (complementary ∠s) $= 30^0$
∠OMT $= 90^0$
But ∠OMR $= 60^0$ (base ∠s of isosceles ΔOMR)
Therefore, $x = 90^0 - 60^0$
$= 30^0$

Example 14

Figure 7.35 represents a circle centre O and radius 5 cm. The tangents PA and PC are each 12 cm long. Find:
(a) OP (b) ∠APC

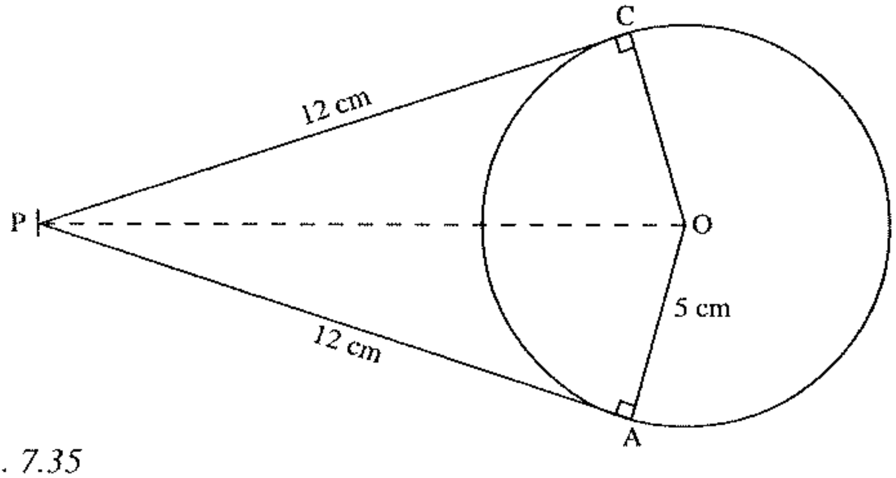


Fig. 7.35

Solution

(a) Join O to P
$OP^2 = OC^2 + PC^2$ (Pythagoras' theorem)
$OP^2 = 5^2 + 12^2$
$= 25 + 144$
$= 169$
Therefore, OP = 13 cm

(b) ∠APC = 2 ∠APO (PO bisects ∠APC).
∠PAO $= 90^0$ (tangent ⊥ radius)
∴ ΔAPO is right-angled at A
$\cos \angle APO = \frac{12}{13}$
$= 0.9231$
Therefore, ∠APO $= 22.62^0$
Hence ∠APC $= 2 \times 22.62$
$= 45.24^0$

Exercise 7.3

1. Figure 7.36 shows a circle centre O. AP is a tangent to the circle at A.

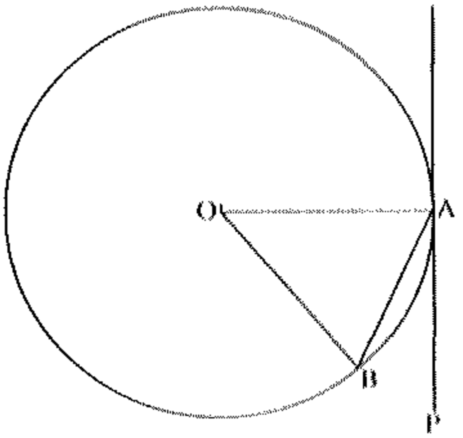


Fig. 7.36

(a) Find ∠OAB amd ∠AOB, given that ∠BAP = 20^0.
(b) Find ∠OAB and ∠BAP given that ∠AOB is 56^0.
(c) Calculate OP, given that OA = 3 cm and PA = 4 cm.
(d) Calculate AP, given that OB = 5 cm and OP = 13 cm.

2. The distance from a point X to the centre of a circle is 10 cm. If the radius of the circle is 6 cm, calculate the length of the tangent from X to the point of contact with circle.
3. In figure 7.37 ∠APB = 42^0. Calculate ∠AOB.

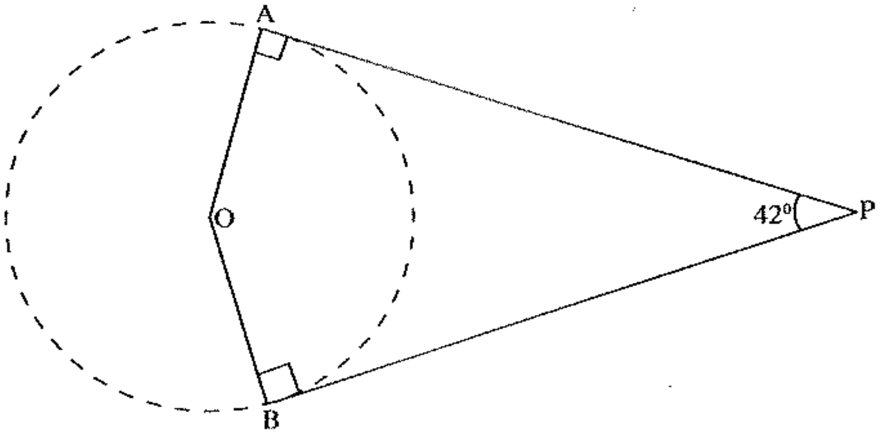


Fig. 7.37

4. In figure 7.37 ∠SPQ = 78^0, ∠PQR = 71^0 and ∠PSR = 104^0. Calculate the angles of the cyclic quadrilateral KLMN.

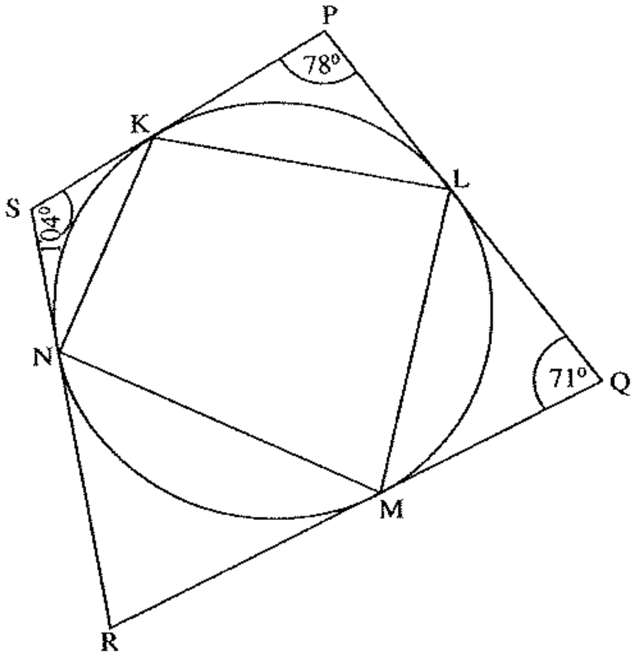


Fig. 7.38

5. The chord PQ subtends an angle of 67^0 at the centre of a circle. Find the size of the obtuse angle between PQ and the tangent to the circle at Q.
6. Figure 7.39 shows two concentric circles with radii 12 cm and 13 cm respectively. If RS is a tangent to the inner circle, calculate its length.

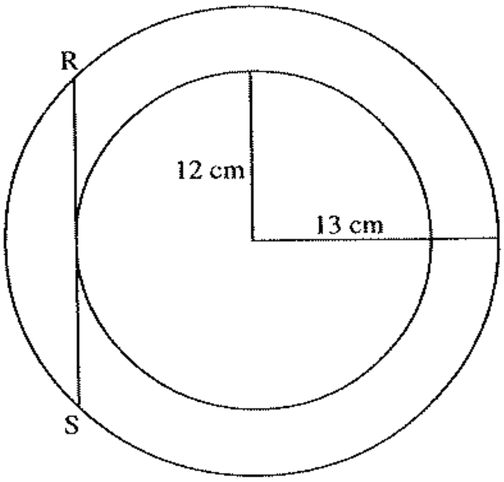


Fig. 7.39

7. In figure 7.40 QT and RT are tangents to a circle. If $\angle QPR = 40^0$, find X.

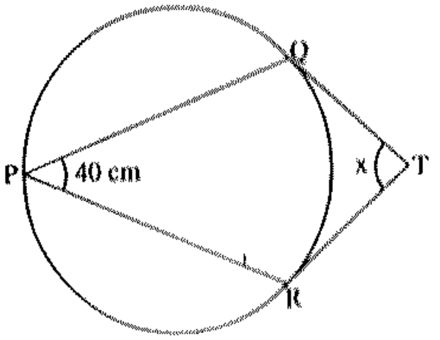


Fig. 7.40

8. In figure 7.41 A and B are centres of the two circles which touch externally at T. If PR is a common tangent to both circles, AP = 17 cm, PB = 10 cm, and PT = 8 cm, calculate AB.

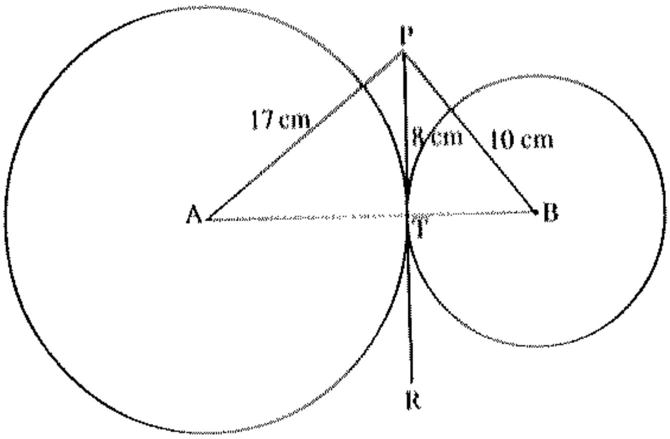


Fig. 7.41

9. In figure 7.41 PT and QT are tangents to the circle centre O. If QT is produced to R and ∠POT = x, show that ∠PTR = 2x.

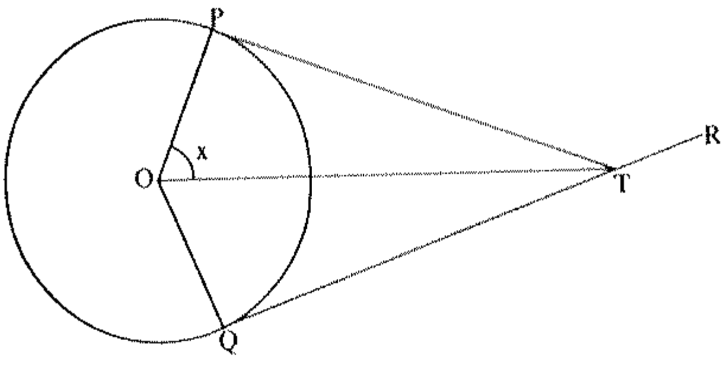


Fig. 7.42

10. In figure 7.43 RP, RQ and QP are tangents to the given circle at A, B and C respectively. If $\angle RPQ$ is 56^0 and $\angle RQP$ is 60^0, calculate:
    (a) $\angle BAC$.
    (b) $\angle ACB$.
    (c) $\angle ABC$.

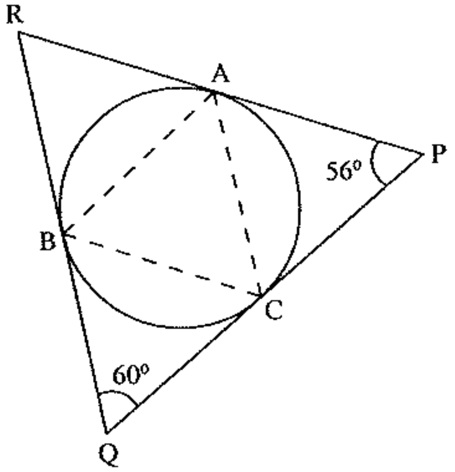


Fig. 7.43

11. Figure 7.44 shows a circle centre O in which BA and BX are tangents. Given that $\angle OCD = 30^0$, AB = 4 cm and DB = $2\sqrt{3}$ cm, find:
    (a) x and y.
    (b) AD.

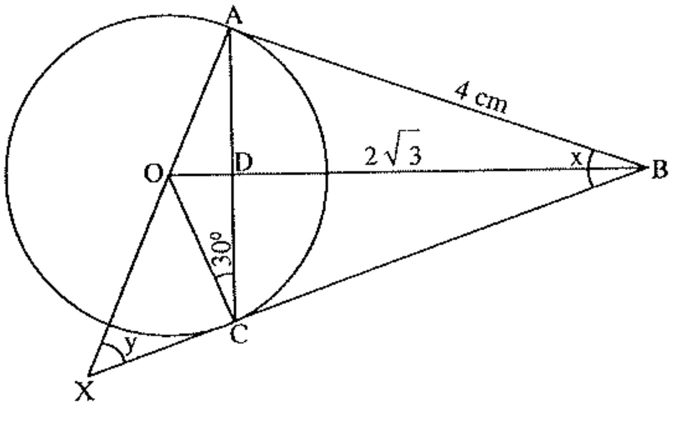


Fig. 7.44

7.4: Tangents to Two Circles

Direct (Exterior) common tangents

(a) In figure 7.45 P and Q are centres of two circles with radii r_1 and r_2 respectively. Given that $r_1 > r_2$, construct the exterior common tangents to both circles.

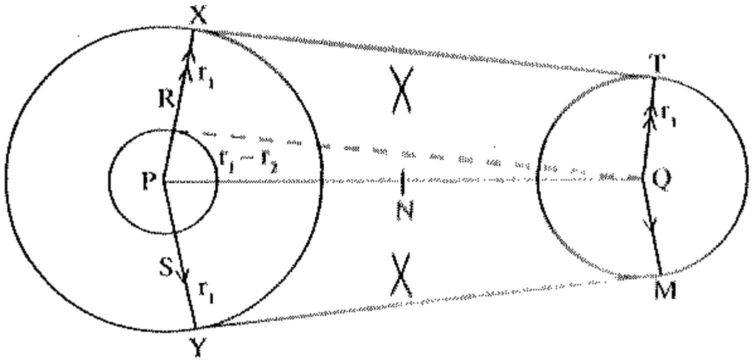


Fig. 7.45

Procedure

(i) With centre P, construct a circle whose radius is $r_1 - r_2$.

(ii) Join P to Q.

(iii) Bisect PQ to get N.

(iv) With centre N and radius PN, draw arcs to cut the circle whose radius is $r_1 - r_2$ at R and S. Join Q and R and Q to S.

(v) Produce PR and PS to cut the circle whose radius is r_1 at X and Y respectively.

(vi) Draw QT parallel to PX and QM parallel to PY.

(vii) Join Y to M and X to T. These are the required direct (exterior) common tangents.

Note:

$PR = r_1 - r_2$ (construction)

$PX = r_1$ (given)

$\therefore RX = r_1 - (r_1 - r_2)$

$= r_2$

$\therefore RX = QT$.

But RX is parallel to QT (construction).

$\therefore$ RQTX is a parallelogram (opposite sides equal and parallel).

QR is tangent to circle radius PR (construction).

$\angle PRQ = 90^0$ (radius $\perp$ tangent).

$\therefore \angle XRQ = 90^0$ (adjacent $\angle$s on a straight line PX).

$\therefore \angle RQT = 90^0$ (interior $\angle$s of parallelogram).

Thus, RQTX is a rectangle.

Therefore, $\angle RXT = 90^0$ and $\angle QTX = 90^0$

Hence, XT is a common tangent to both circles at points X and T.

Similarly, it can be shown that YM is a common tangent to both circles at points Y and M.

Transverse (Interior) Common Tangents

In figure 7.46, P and Q are centres of two circles with radii r_1 and r_2 respectively. Given that $r_1 > r_2$, construct the transverse common tangents to both circles.

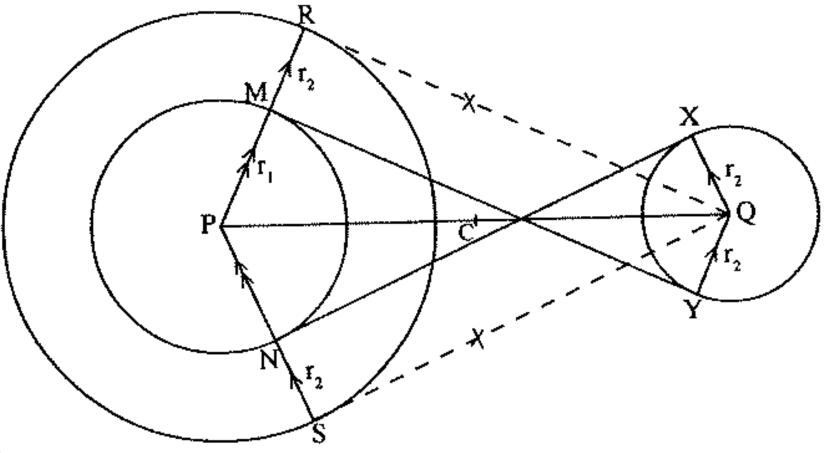


Fig. 7.46

Procedure

(i) With centre P, construct a circle whose radius PR is equal to $r_1 + r_2$.

(ii) Join P to Q and bisect PQ to get point C.

(iii) With centre C and radius PC, draw arcs to cut the circle whose radius is $r_1 + r_2$ at R and S. Join Q to R and Q to S.

(iv) Draw the lines PR and PS to cut the circle whose radius is r_1 at M and N respectively.

(v) Draw line QX parallel to PS and line QY parallel to PR.

(vi) Draw lines MY and NX. These are the required transverse (interior) common tangents.

Note:

$PR = r_1 + r_2$ (construction).

$PM = r_1$ (given)

$\therefore RM = PR - PM = (r_1 + r_2) - r_1$

$= r_2$

$\therefore RM = QY$

But RM is parallel to QY (construction)

$\therefore$ MRQY is a parallelogram (opposite sides equal and parallel).

QR is tangent to circle radius PR (construction).

$\angle PRQ = 90^0$ (radius $\perp$ tangent).

But $\angle YQR = 90^0$ (interior $\angle$s of a parallelogram).

$\therefore$ MRQY is a rectangle.

$\therefore \angle PMY = \angle QYM = 90^0$.

Thus, MY is the transverse tangent to both circles at points M and Y. Similarly, it can be shown that NX is another transverse tangent to both circles at points N and X.

Exercise 7.4

1. Draw a circle centre O radius 3 cm. If P is a point 7 cm from O, construct two tangents to the circle from P to touch the circle at points Q and R. Measure QP.
2. Draw two circles radii 2 cm and 3 cm with centres 6 cm apart. Construct an exterior common tangent to the circles and measure its length.
3. Draw two equal circles of radii 2 cm with their centres 7 cm apart and construct the transverse common tangents to the circles.
4. Figure 7.47 shows two circles centres P and Q radii 6 cm and 1 cm respectively. The distance between their centres is 8 cm. Find by construction the distance RS and the size of $\angle QPS$.

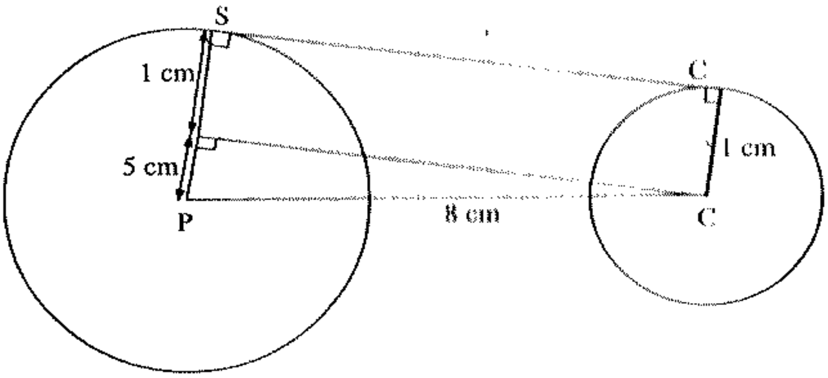
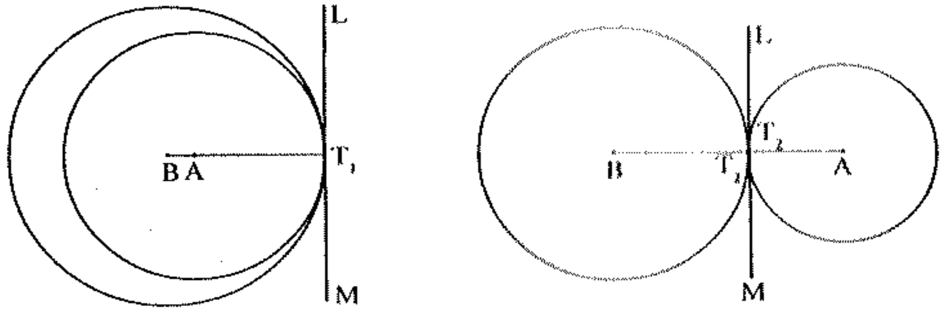


Fig. 7.47

7.5: Contact of Circles

Two circles are said to touch each other at a point if they have a common tangent at that point.

In figure 7.48 (a), the two circles with centres A and B touch each other **internally** at T_1, while in figure 7.47 (b), they touch each other **externally** at T_2.



(a) (b)

Fig. 7.48

Note:

(i) The centres of the two circles and their point of contact lie on a straight line.

(ii) When two circles touch other internally, as in figure 7.48 (a), the distance between the centres is equal to the difference of the radii, i.e. $BA = BT_1 - AT_1$.

(iii) When two circles touch each other externally as in figure 7.48 (b) the distance between the centres is equal to the sum of the radii, i.e.,
$AB = AT_2 + BT_2$

Example 15

Figure 7.49 shows circles P, Q and R with centres C, B and A respectively.

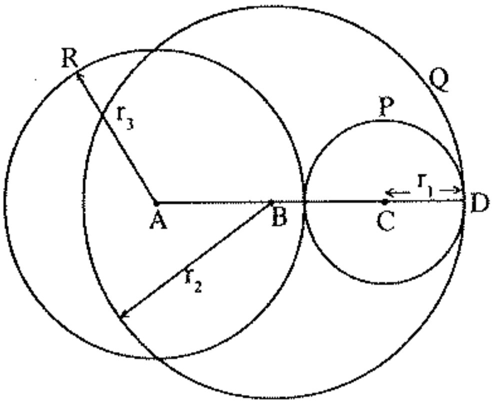


Fig. 7.49

Given that the radius of P is r_1, that of Q is r_2 and that of R is r_3, find in terms of r_1, r_2 and r_3:

(a) AD (b) BC (c) AC

Solution

(a) $AD = r_3 + 2r_1$

(b) $BC = r_2 - r_1$

(c) $AC = r_3 + r_1$

Example 16

Figure 7.50 shows the tops of two circular tins in contact at T. The tins are of equal height but different radii. Their centres are X and Y. A is a point on their common tangent such that AY = 15 cm, AX = 13 cm and AT = 12 cm. Calculate the distance between their centres.

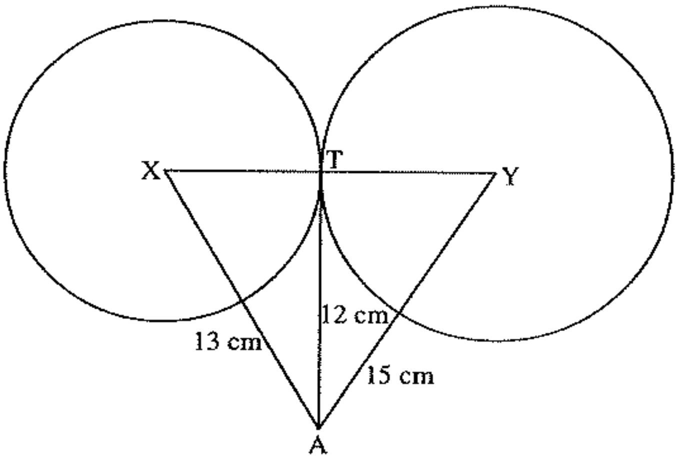


Fig. 7.49

Solution

From the figure;

$\angle ATX = 90^0$ (radius ⊥ to tangent)

$\angle ATY = 90^0$ (radius ⊥ to tangent)

$\therefore$ Triangles ATX and ATY are right-angled at T.

From tringle ATX;

$XT^2 = 13^2 - 12^2$ (Pythagoras' theorem)

$= 25$

$XT = \pm 5$

Hence, XT = 5 cm.

From triangle ATY;

$YT^2 = 15^2 - 12^2$ (Pythagoras' theorem)

$= 81$

$YT = \pm 9$

Hence, YT = 9 cm

$\therefore XY = 5 + 9$

$= 14$ cm

Example 17

PQR is a hemispherical bowl which contains two metal spheres. The centre of the bowl is Z and those of the spheres are X and Y as shown in figure 7.51. Given that XY = 16 cm, XZ = 17 cm and YZ = 15 cm, calculate the radii of the three spheres.

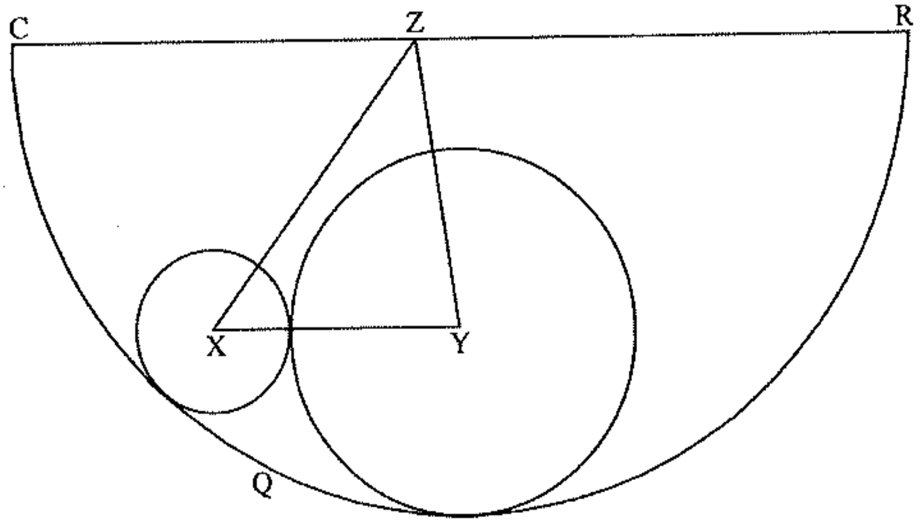


Fig. 7.51

Solution

Let r_1, r_2 and r_3 be the radii of the spheres with centres at X, Y and Z respectively.

From X to Y, $r_1 + r_2 = 16$ cm ....(1)

From Z to Y, $r_3 - r_2 = 15$ cm ....(2)

From Z to X, $r_3 - r_1 = 17$ cm ....(3)

Adding (1) and (2);

$r_3 + r_1 = 31$ cm ....(4)

Adding (3) and (4),

$2r_3 = 48$ cm

$r_3 = 24$ cm

$r_2 = 9$ cm (from (ii))

$r_1 = 7$ cm (from (i))

$\therefore$ The radius of the bowl is 24 cm while that of the spheres with centres X and Y are 7 cm and 9 cm respectively.

Exercise 7.5

1. Figure 7.52 shows four equal cylindrical rods tied together with a taut string such that PQRS forms a square:

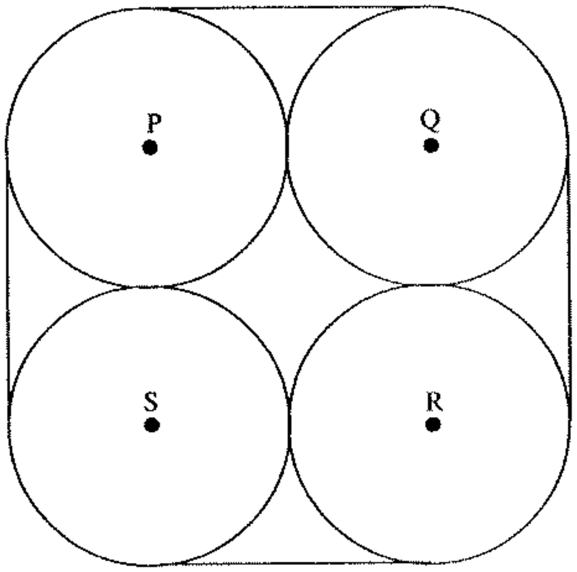


Fig. 7.52

If PQ is 28 cm, calculate the length of the string. (Take $\pi = \frac{22}{7}$)

2. Two circles radii 3 cm and r cm touch each other externally at T. If the length of their common tangent PT is 8 cm, calculate the value of r.
3. Figure 7.53 shows two circles centres O_1 and O_2 touching each other internally at point T. PT is the common tangent.
   If PT = 24 cm, PO_2 = 26 cm and PO_1 = 30 cm, calculate the length O_1O_2.

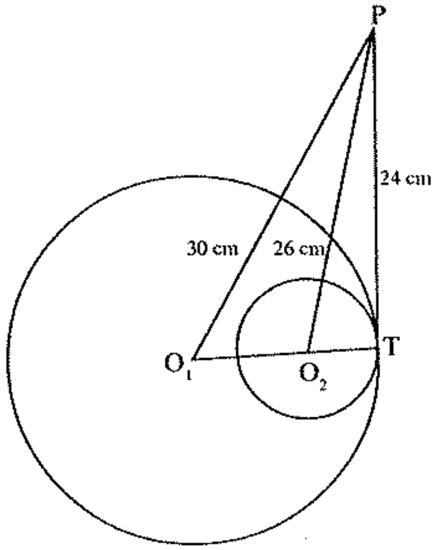


Fig. 7.53

4. Figure 7.54 shows two circles centres O_1 and O_2 touching each other externally at point T. PT is the common tangent. If the ratio of their radii is 3 : 2, $PO_1 = 50$ cm and $PO_2 = 20\sqrt{5}$ cm, calculate O_1O_2.

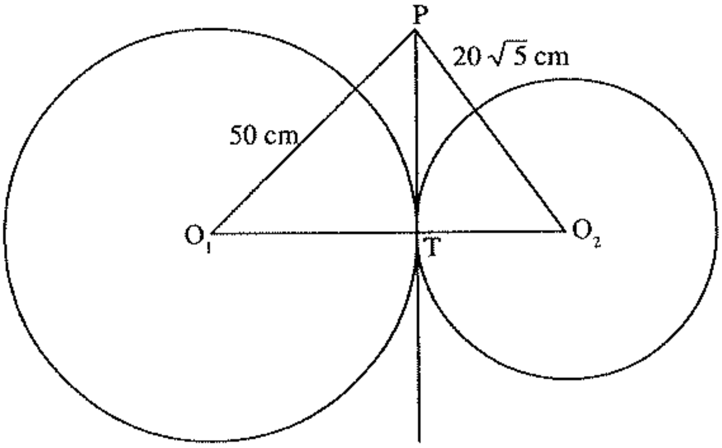


Fig. 7.54

5. Two circles with centres P and Q touch internally at X. R is a point on their common tangent such that RX = 8 cm, RQ = 10 cm and RP = 16 cm.
   (a) Calculate PQ.
   (b) If the circles touch externally instead of internally, find PQ.
6. In figure 7.55, the circles centres O and C touch internally at N and ∠POQ = 60°.

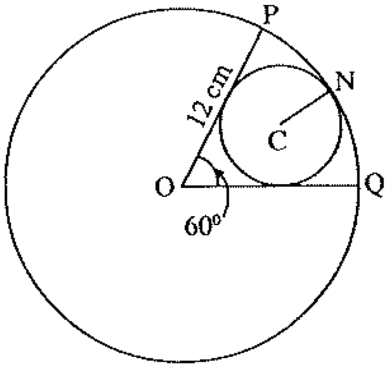


Fig. 7.55

If the radius of the larger circle is 12 cm, calculate the radius of the smaller circle.

7. Figure 7.56 shows two circles centres O_1 and O_2 touching externally at T. P is a point on their common tangent. Tangents from P touch one of the circles at R and the other at S. If PR = 8 cm, $PO_2 = 10$ cm and $PO_1 = 17$ cm,

find:
(a) PT
(b) PS
(c) O_1O_2

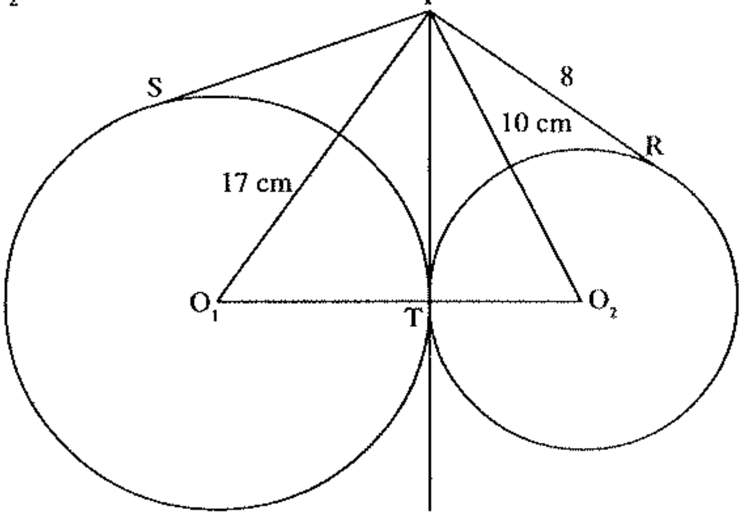


Fig. 7.56

8. Figure 7.57 shows two concentric circles with centres at O and radii 16 cm and 20 cm. Calculate the length of AB if it is a tangent to the inner circle.

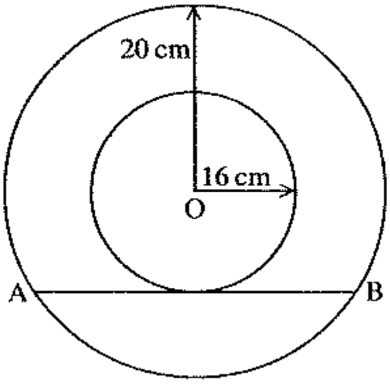


Fig. 7.57

9. The centres of three circles form a triangle PQR in which PQ = 8 cm, QR = 10 cm and PR = 12 cm. If the circles are such that each touches the other two externally, find the radii of the circles.

7.6: Angle in Alternate Segment

In figure 7.58, RPT is a tangent to the circle at P and the chord PQ divides the circle into two segments PBQ and QAP:

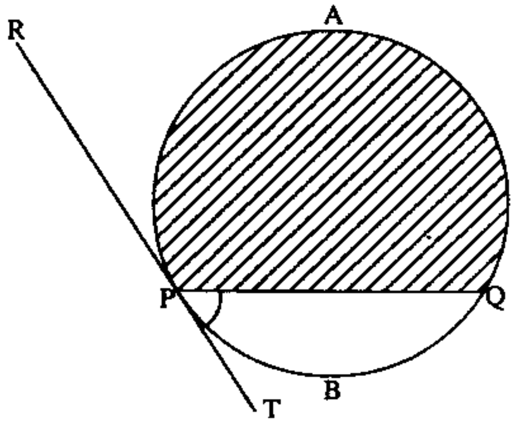


Fig. 7.58

We say that the segment PAQ (shaded) is **alternate** to angle TPQ because it is on the side of PQ which is opposite to angle TPQ. Similarly, the segment PBQ (unshaded) is alternate to angle RPQ. In figure 7.59, angle SRT lies in an alternate segment to angle STX, where XTY is a tangent to the circle.

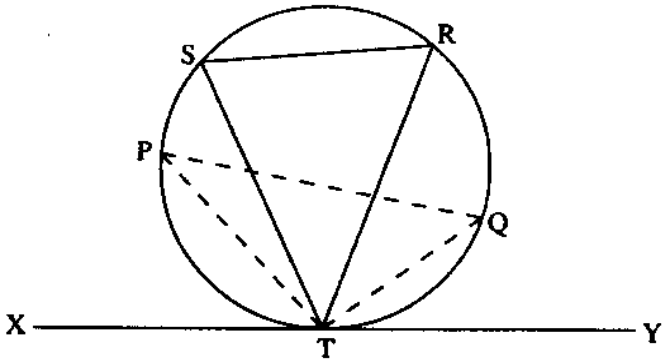
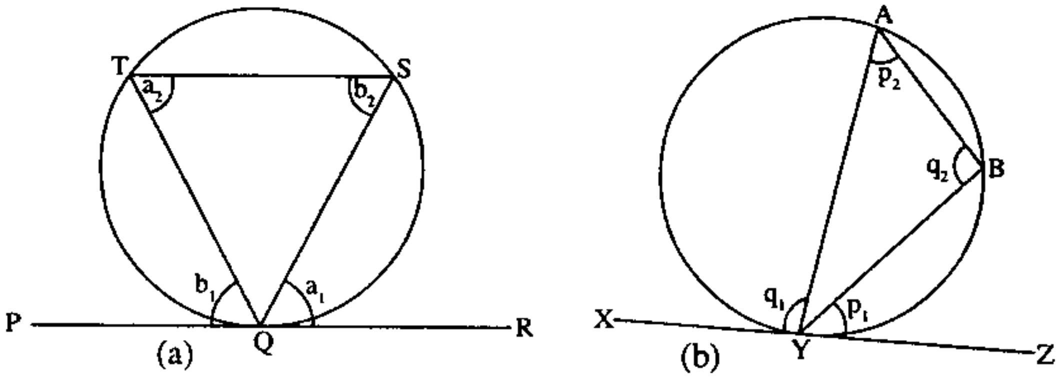


Fig. 7.59

Similarly, angle RST lies in an alternate segment to angle RTY.

Draw two circles of any radius as shown in figure 7.60 (a) and (b). Draw tangents PQR and XYZ accurately.



(a) (b)

Fig. 7.60

Measure the following pairs of angles:
a_1 and a_2.
b_1 and b_2.
q_1 and q_2.
and p_1 and p_2
Consider figure 7.61.

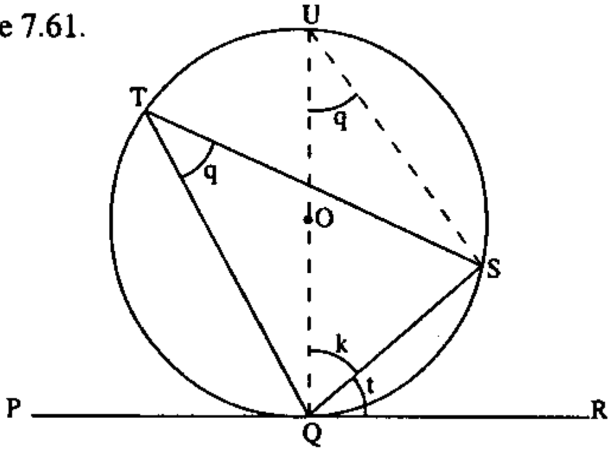


Fig. 7.61

PQR is a tangent to the circle TQS at Q. O is the centre of the circle. Show that the acute angle t is equal to angle q.

Construct QU to pass through O. Join U to S.
$\angle QTS = \angle QUS = q$ ($\angle$s in same segment).
$t + k = 90^0$ (tangent $\perp$ radius)
$\angle USQ = 90^0$ (angle in a semicircle).
$k + q = 90^0$ (complementary $\angle$s)
Hence, $k + q = t + k = 90^0$.
Therefore, $q = t$
Consider figure 7.62.

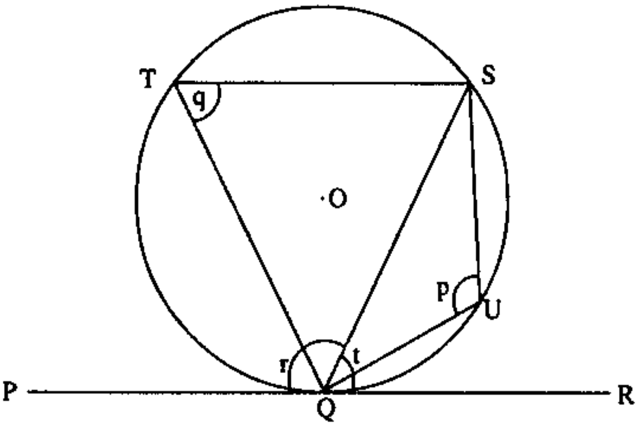


Fig. 7.62

PQR is a tangent to the circle TQUS at Q. Show that the obtuse angle r is equal to angle p.

Solution

q + p = 180^0 (opp. ∠s of cyclic quad.)
r + t = 180^0 (adj. ∠s on a straight line)
Hence, q + p = r + t = 180^0
But t = q (angle in alt. seg.)
Therefore, p = r

The results obtained may be stated as follows:

If for any tangent to a circle a chord is drawn from the point of contact, the angle which the chord makes with the tangent is equal to the angle subtended by the same chord in the alternate segment of the circle.

Example 18

In figure 7.63, PQR and PAX are tangents to a circle centre O. B is a point on the major arc QA.

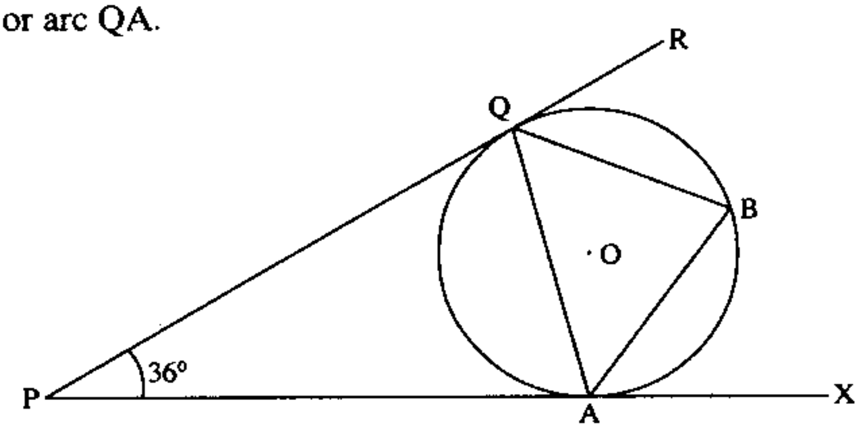


Fig. 7.63

If ∠QPA = 36^0, calculate ∠QBA.

Solution

In ∠PQA, PQ = PA (tangents from an external point).
∠PAQ = ∠PQA (base angles of isosceles triangle).
∠QPA = 36^0 (given)
Therefore, ∠PAQ = ∠PQA

$$= \frac{180^0 - 36^0}{2}$$

$$= 72^0$$

∠PAQ = ∠QBA (angles in alternate segments are equal).
Therefore, ∠QBA = 72^0.

Example 19

In figure 7.64, ABP is a secant and PQR is a tangent to the circle at Q.
Show that PA × PB = PQ^2.

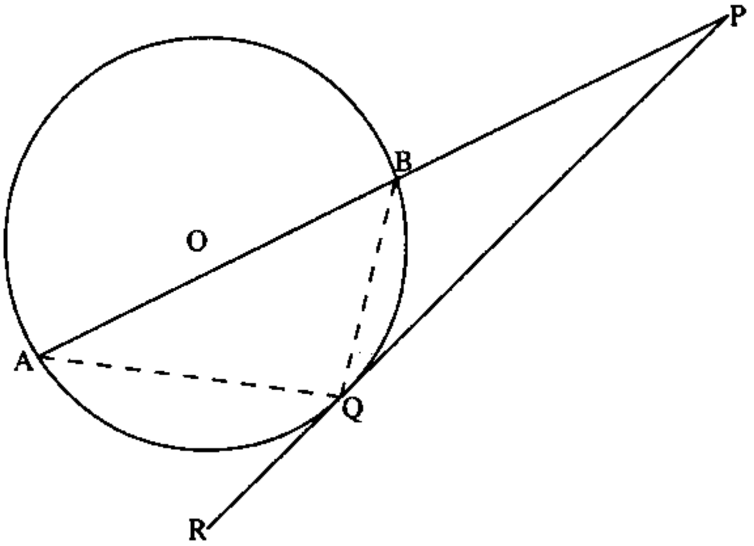


Fig. 7.64

Solution

Join A to Q and Q to B.

In the Δs APQ and QPB;

∠QAP = ∠BQP (angles in alternate segment).

∠APQ = ∠QPB (common angle).

Therefore, the third angles AQP and QBP of the triangles are equal.

So, Δs APQ and QPB are similar.

Therefore $\frac{PA}{PQ} = \frac{PQ}{PB}$ (corr. sides proportional)

Thus, PA.PB = PQ^2

Exercise 7.6

1. Find the value of the angles marked a and b in each of the following figures where O is the centre of each circle.

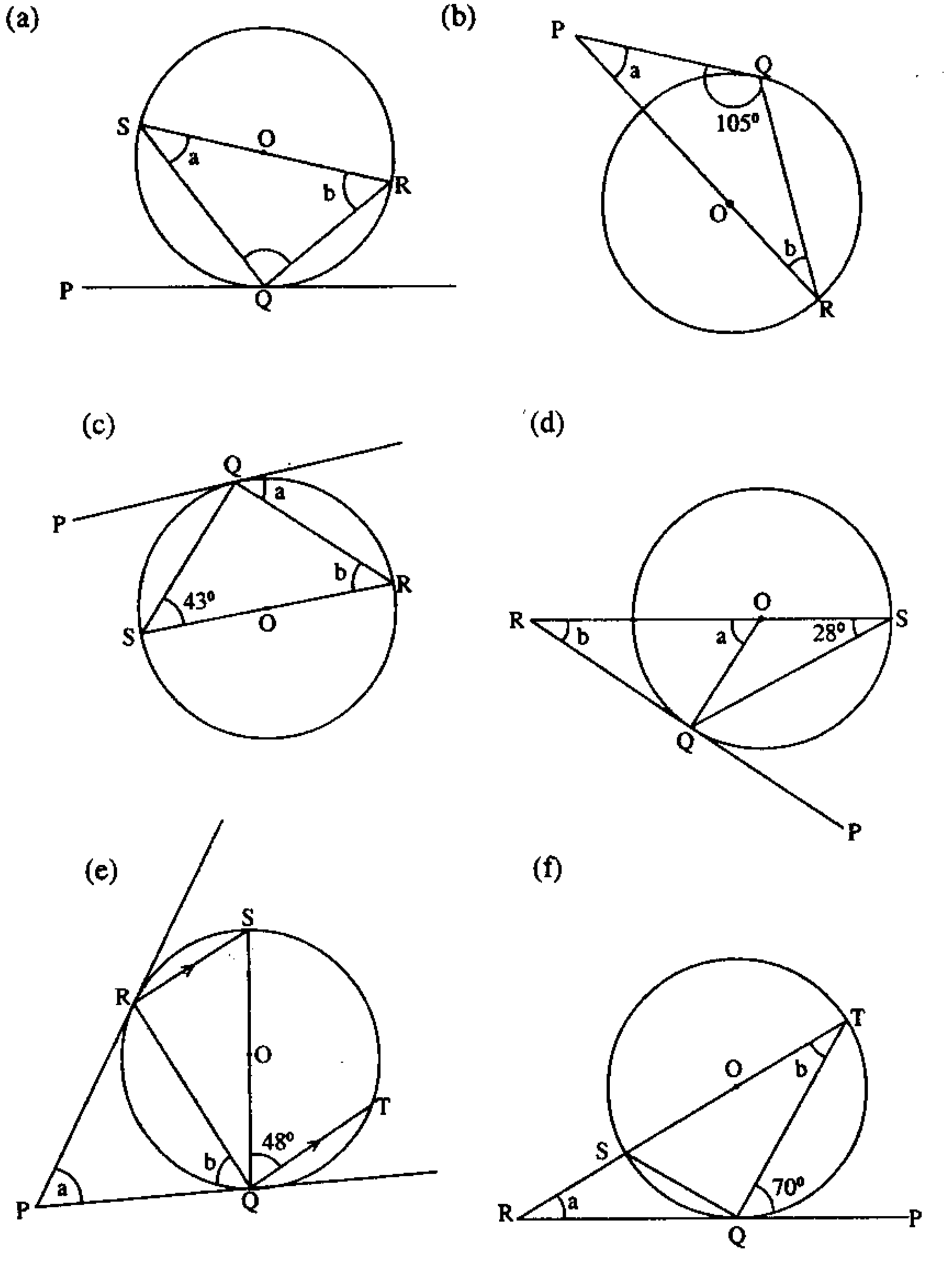


Fig. 7.65

2. In figure 7.66 below, ∠DCF = 50° and ∠ACB = 40°. Find ∠BCE and ∠ABC.

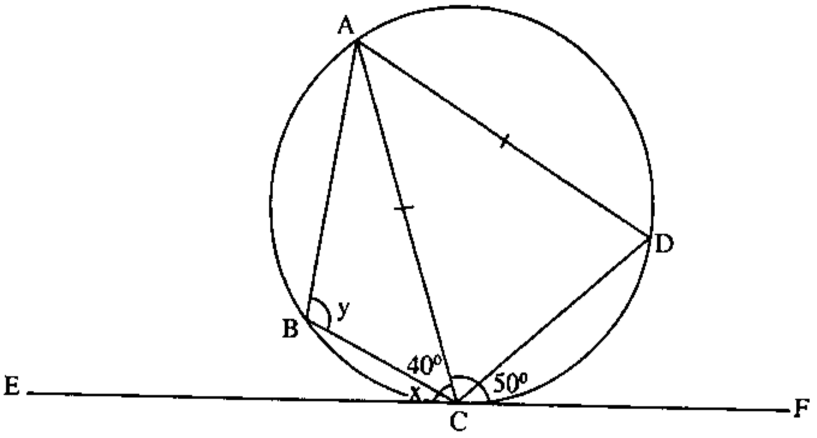


Fig. 7.66

3. In figure 7.67, AB = 5 cm and CD = 7 cm. Find BC.

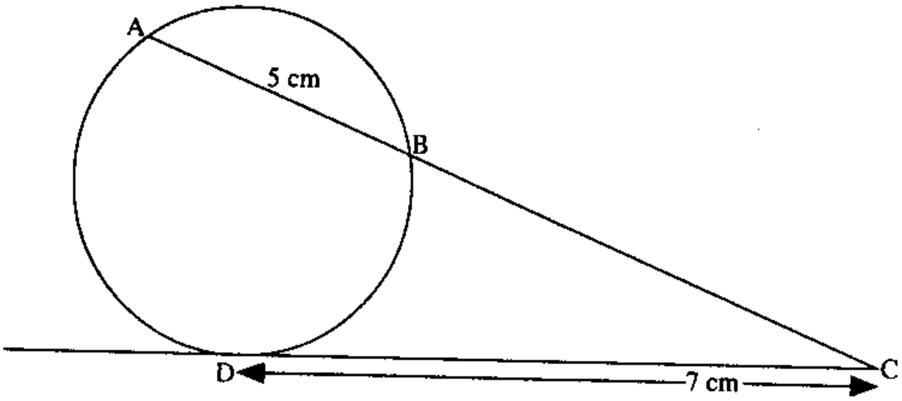


Fig. 7.67

4. In figure 7.68, find x and y.

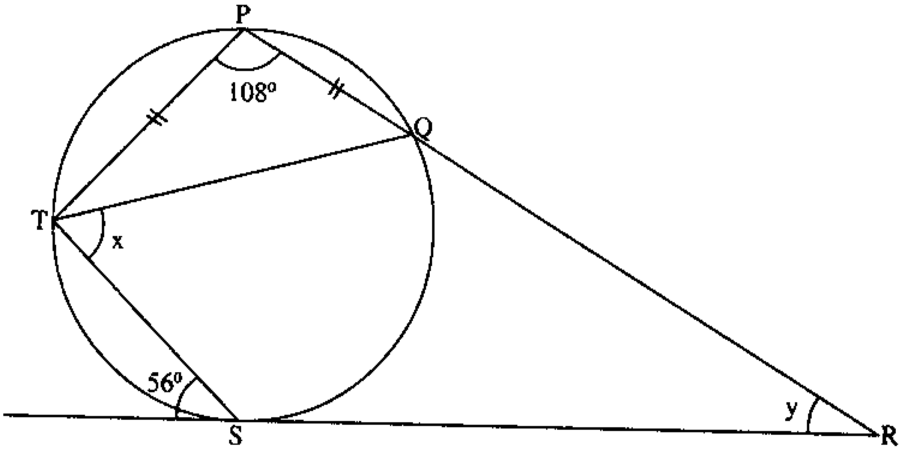


Fig. 7.68

5. In figure 7.69, TDX is a tangent, ∠CDX = 66°, ∠ADT = 62° and ∠BAC = 22°. Find the angles of Δs ABD, ACD and BCD.

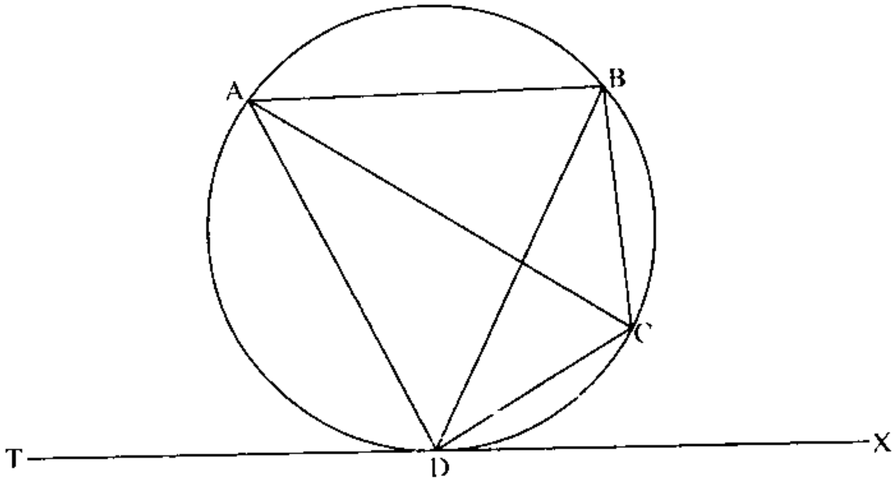


Fig. 7.69

6. In figure 7.70, ABX is a tangent, ∠CAB = 17° and ∠ACB = 36°. Calculate ∠CBX and ∠DBC.

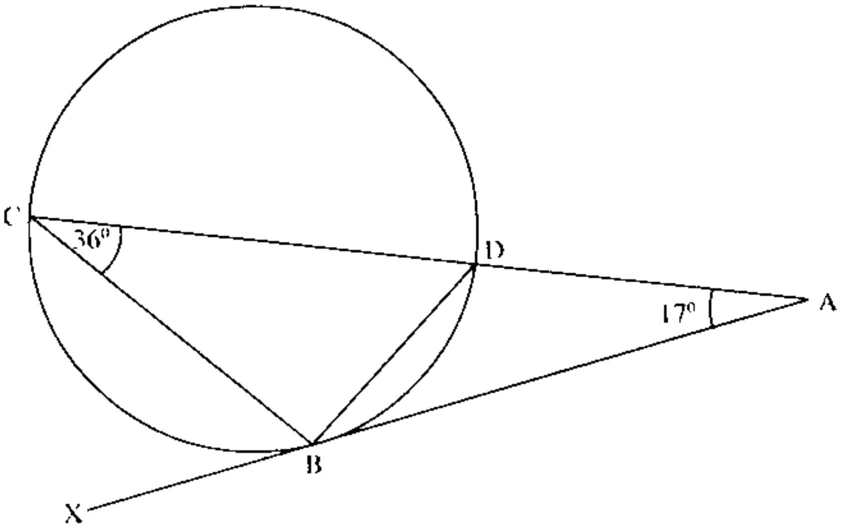


Fig. 7.70

7. TX and TY are tangents to a circle from an external point T. XY is a chord of the circle. Z is a point on the minor arc XY. If ∠XTY = 60° and ∠ZYT = 30°, find ∠TXZ.

7.6: Circles and Triangles

Inscribed Circle

Construct any triangle ABC. Construct the bisectors of the three angles. The bisectors will meet at a point O, as in figure 7.71.

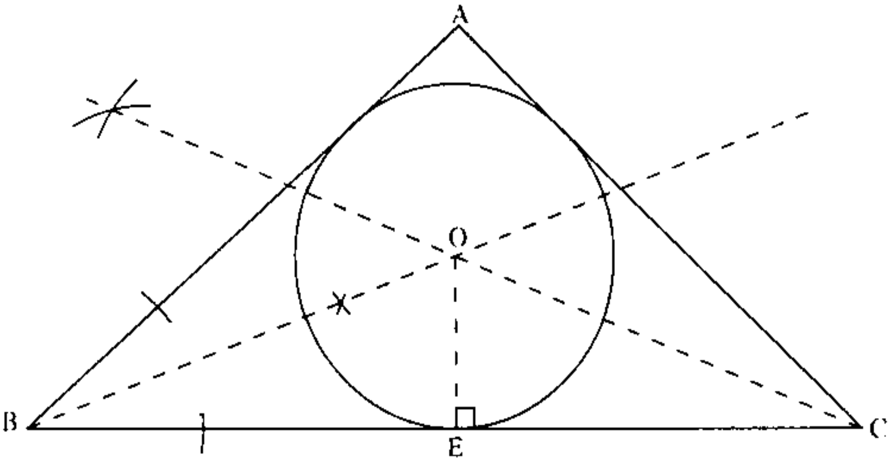


Fig. 7.71

Construct a perpendicular from O to meet one of the sides at E. with centre O and radius OE draw a circle. The circle will touch the three sides of the triangle ABC. Such a circle is called an **inscribed circle** or **incircle**. The centre of an inscribed circle is called the **incentre**.

Circumscribed Circle

Construct any triangle ABC. Construct perpendicular bisectors of AB, BC and AC to meet at point O. With O as centre and using either OA or OB or OC as radius, draw a circle. The circle will pass through the vertices A, B and C. as in figure 7.72.

discuss

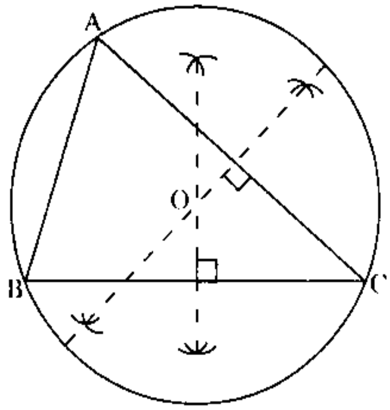


Fig. 7.72

Such a circle is called a **circumcircle** of a triangle and O the **circumcentre**.

Escribed Circles

Construct a triangle ABC. Let D and E be points on AB and AC produced respectively, as shown in figure 7.753.

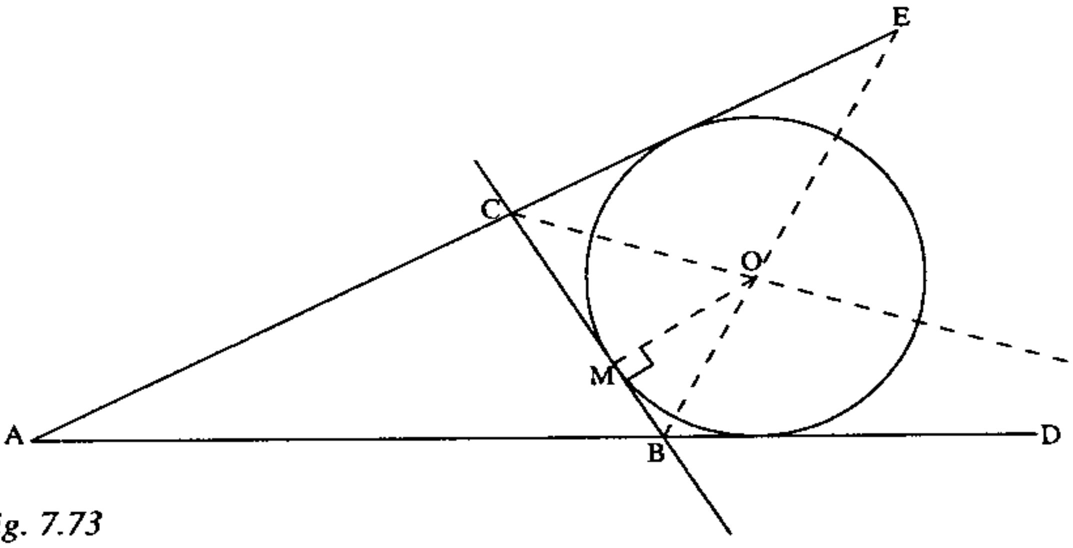


Fig. 7.73

Construct the bisectors of the external angles CBD and BCE to meet at O. Construct a perpendicular OM to meet BC at M. With O as centre and radius OM, draw a circle. The circle will touch each of the sides of the triangle ABC externally. Such a circle is called an escribed circle and the centre O is the **ex-centre**. BO and CO are called **external bisectors** of angles ABC and ACB respectively.

Note:
By producing BA and BC, CA and CB, two more escribed circles for the same triangle can be constructed.

Other Centres of Triangle (Centroid and Orthocentre)

(i) Construct any triangle ABC. Draw altitudes AM, BN and CL as in figure 7.74(a). The point of intersection O of the altitudes is called the **orthocentre**.

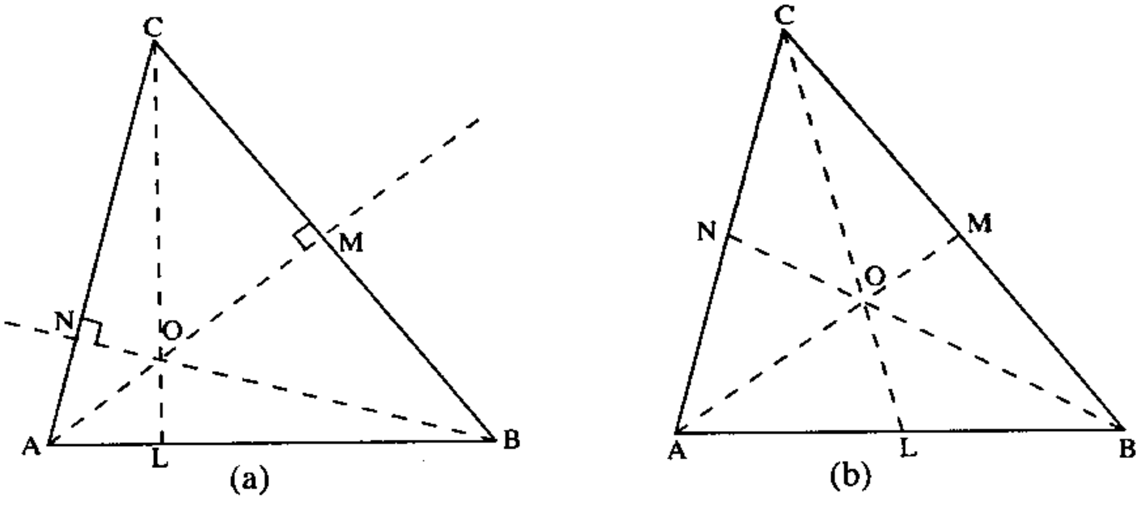


Fig. 7.74

(ii) Draw triangle ABC. Draw **medians** AM, BN and CL, as in figure 7.74(b). The point of intersection O of the medians is called the **centroid** of triangle ABC.

***Exercise** 7.7*

1. Construct an equilateral triangle ABC of side 6 cm. Draw the inscribed circle and measure its radius.
2. Construct an equilateral triangle ABC of side 4 cm. Draw the escribed circles and measure their radii.
3. Construct ΔPQR with PQ = 5.8 cm, QR = 3.4 cm and PR = 4.1 cm. Draw a circle passing through P, Q and R. Measure its radius.
4. Construct ΔXYZ with XY = 6.0 cm, XZ = 5.1 cm and ∠YXZ = 50^0. By construction, find the radius of the inscribed circle.
5. Given triangle ABC with BC = 6 cm, AB = 8 cm and ∠ABC = 90^0, locate the orthocentre and measure AC.
6. Construct a right isosceles triangle KLM whose base KL = 8 cm. Measure the length between the incentre and the circumcentre.
7. In ΔKLM, KL = 4 cm, KM = 2.5 cm and angle MKL = 50^0. Determine the ex-centres of the escribed circles touching LK and LM.
8. In ΔABC, AB = 4 cm, BC = 3 cm and ∠ABC = 90^0. Locate the three excentres and draw the ex-circles.
9. Draw an isosceles triangle ABC with the base angles of 40^0 and AB = AC = 8 cm.
   (a) Locate:
      (i) the centroid C.
      (ii) the circumcentre O.
   (b) Draw the circumcircle and measure:
      (i) circumradius,
      (ii) OC.
10. Draw a triangle PQR with PQ = 8 cm, QR = 7 cm and RP = 6 cm. Locate the orthocentre C and measure CQ.

Chapter Eight

MATRICES

8.1: Introduction

Certain types of information are best presented in form of tables. Below are some examples.

(i) Some premier league results in second week of April 2000:

Team	*Played*	*Won*	*Drawn*	*Lost*	*Points*
Tusker	8	5	3	0	18
Mumias	7	5	2	0	17
Utalii	8	5	1	2	16
Ulinzi	7	4	2	1	14
Mathare United	8	3	4	1	13

(ii) Ramadhan timetable for 21st November 2002:

Town	*Closing (p.m.)*	*Opening (a.m.)*
Nairobi	5.00	6.39
Mombasa	4.51	6.24
Nakuru	5.02	6.43
Kisumu	5.07	6.48

(iii) Recommended dosage for treatment of malaria.

Day	*Adults* *Over (15 yrs)*	*(11 – 15 yrs)*	*(6 – 10 yrs)*
First day	4	3	2
After six hours	2	1	1
Second day	2	1	1
Third day	2	0	0

If what each row and column represents is understood, the headings can be omitted and the information represented as follows:

(a) The national football league table becomes:

$$\begin{pmatrix} 8 & 5 & 3 & 0 & 18 \\ 7 & 5 & 2 & 0 & 17 \\ 8 & 5 & 1 & 2 & 16 \\ 7 & 4 & 2 & 1 & 14 \\ 8 & 3 & 4 & 1 & 13 \end{pmatrix}$$

(b) The Ramadhan timetable becomes:

$$\begin{pmatrix} 5.00 & 6.99 \\ 4.51 & 6.24 \\ 5.02 & 6.43 \\ 5.07 & 6.48 \end{pmatrix}$$

(c) The recommended dosage becomes:

$$\begin{pmatrix} 4 & 3 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

Each is a rectangular pattern of numbers. Such a pattern (array) is called a **matrix** (*plural*, **matrices**). A matrix is always enclosed in brackets.

8.2: Order of a Matrix

A matrix consists of rows and columns. Rows are the horizontal arrangement while column are the vertical arrangement. The dosage matrix above consists of four rows and three columns.

The number of rows and columns determine the order of a matrix, which is given by stating the number of rows followed by the number of columns. The dosage matrix, is for example, of order 4 × 3 and, read as 'four by three'. State the order of the following matrixes:

(i) The football league matrix above.

(ii) The Ramadhan timetable above.

(iii) $\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$ (iv) $\begin{pmatrix} 5 & 1 & 7 \\ 0 & 4 & 2 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 4 & 3 & 9 & 5 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 & 2 \\ 4 & 5 \\ 3 & 7 \\ 1 & 9 \end{pmatrix}$ (vii) $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

In general, if the number of rows is m and the number of columns n, the matrix is of order m x n.

Elements of a Matrix

Each number or letter in a matrix is called an element of the matrix. Each element can be located in the matrix by stating its position in the row and the column.

For example, given the 2 × 2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

a_{11} means the element in the first row and first column.

a_{12} means the element in the first row and second column.

a_{21} means the element in the second row and first column.
a_{22} means the element in the second row and second column.

In general a^{ij} means an element in the i^{th} row and j^{th} column.

A matrix in which the number of rows is equal to the number of columns is called a **square** matrix.

The football league matrix is a square matrix of order 5 × 5. A matrix with only one row like (1 4 3 9 5) is called a row matrix.

A matrix with only one column like $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ is called a column matrix.

A matrix with all elements zero is called zero or null matrix. Matrices are denoted by a capital letter in bold face, as shown below:

$\mathbf{A} = \begin{pmatrix} 2 & 1 & 6 \\ 3 & 3 & 7 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 5 & -3 \\ 4 & 8 \end{pmatrix}$

Two or more matrices are equal if they are of the same order and their corresponding elements are equal.

Thus, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix}$ then, $a = 3, b = 4, c = 1$ and $d = 5$

8.3: Addition and Subtraction of Matrices

Matrices can be added or subtracted if they are of the same order. The sum of two or more matrices is obtained by adding corresponding elements. Subtraction is also done in the same way.

Example 1

If $\mathbf{P} = \begin{pmatrix} 2 & 5 & 4 \\ 0 & 7 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 3 & 8 \\ 6 & 2 & 5 \end{pmatrix}$, find:

(a) $\mathbf{P} + \mathbf{Q}$ (b) $\mathbf{P} - \mathbf{Q}$

Solution

(a)
$$\begin{aligned} \mathbf{P} + \mathbf{Q} &= \begin{pmatrix} 2 & 5 & 4 \\ 0 & 7 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 8 \\ 6 & 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2+1 & 5+3 & 4+8 \\ 0+6 & 7+2 & 3+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 8 & 12 \\ 6 & 9 & 8 \end{pmatrix} \end{aligned}$$

(b)
$$\begin{aligned} \mathbf{P} - \mathbf{Q} &= \begin{pmatrix} 2 & 5 & 4 \\ 0 & 7 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 8 \\ 6 & 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2-1 & 5-3 & 4-8 \\ 0-6 & 7-2 & 3-5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & -4 \\ -6 & 5 & -2 \end{pmatrix} \end{aligned}$$

Example 2

If $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 0 \\ 5 & 9 & 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 8 & 0 & 2 \\ 1 & 3 & 5 \\ 2 & 1 & 6 \end{pmatrix}$, Find:

(a) $\mathbf{A} - \mathbf{B} + \mathbf{C}$ (b) $\mathbf{A} - (\mathbf{B} + \mathbf{C})$

Solution

(a)
$$\mathbf{A} - \mathbf{B} + \mathbf{C} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 0 \\ 5 & 9 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 2 \\ 1 & 3 & 5 \\ 2 & 1 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 3-2+8 & 2-4+0 & 1-1+2 \\ 0-1+1 & 4-2+3 & 5-0+5 \\ 1-5+2 & 3-9+1 & 2-6+6 \end{pmatrix}$$
$$= \begin{pmatrix} 9 & -2 & 2 \\ 0 & 5 & 10 \\ -2 & -5 & 2 \end{pmatrix}$$

(b)
$$\mathbf{A} - (\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix} - \left[\begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 0 \\ 5 & 9 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 2 \\ 1 & 3 & 5 \\ 2 & 1 & 6 \end{pmatrix}\right]$$
$$= \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 10 & 4 & 3 \\ 2 & 5 & 5 \\ 7 & 10 & 12 \end{pmatrix}$$
$$= \begin{pmatrix} -7 & -2 & -2 \\ -2 & -1 & 0 \\ -6 & -7 & -10 \end{pmatrix}$$

Two or more matrices are compatible for addition or subtraction only if they are of the same order.

Exercise 8.1

1. State the order of each of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ 5 & 2 \\ 1 & 1 \end{pmatrix}$ (d) (5)

(e) $\begin{pmatrix} 4 & 0 & 4 \\ 2 & 2 & 4 \\ 3 & 1 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & -1 & 8 & 7 \\ 0 & 2 & 3 & 5 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 5 & 6 \\ 3 & -2 & 7 \\ 2 & 3 & -1 \\ 4 & 0 & 5 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 4 & 0 & 3 \end{pmatrix}$ (j) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. Give examples of matrices of the following order:

(a) 1 × 3 (b) 2 × 2 (c) 3 × 1

(d) 3 × 3 (e) 2 × 3 (f) 3 × 2

3. Add the following pairs of matrices, where possible:

(a) $\begin{pmatrix} 2 & 1 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 8 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & -4 \\ -3 & 5 \end{pmatrix}, \begin{pmatrix} 12 & 15 \\ -6 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 5 & -2 \end{pmatrix}, \begin{pmatrix} 8 & 11 \\ 6 & 5 \\ 7 & 20 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 5 & -2 \end{pmatrix}, \begin{pmatrix} 8 & -4 \\ -3 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 2.3 & 3.3 & 12.0 \\ 4.1 & 5.6 & 1.8 \\ 8.0 & 7.4 & 11.0 \end{pmatrix}, \begin{pmatrix} 8.3 & 5.2 & 6.1 \\ 0.6 & 9.0 & 5.6 \\ 10.2 & 4.8 & 6.5 \end{pmatrix}$ (f) $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 & 4 \\ 3 & -1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} 4.5 & 7.3 \end{pmatrix}, \begin{pmatrix} 20 & 1 & -2 \end{pmatrix}$

(i) $\begin{pmatrix} 3 & 15 \\ 1 & 16 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 0 \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 5 \end{pmatrix}$

4. Given that $\mathbf{P} = \begin{pmatrix} 14 & 2 \\ 6 & 7 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 5 & 15 \\ 8 & 3 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 6 & 8 & 12 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 12 & 4 & 7 \\ 5 & 16 & -1 \\ 8 & 3 & 4 \end{pmatrix}$,

find, where possible:

(a) $\mathbf{P} - \mathbf{Q}$ (b) $\mathbf{Q} - \mathbf{P}$ (c) $\mathbf{R} - \mathbf{S}$ (d) $\mathbf{S} - \mathbf{R}$

(e) $\mathbf{S} - \mathbf{P}$ (f) $\mathbf{S} - \mathbf{Q}$ (g) $\mathbf{R} - \mathbf{S} - \mathbf{Q}$

5. If $\mathbf{A} = \begin{pmatrix} p & -q & r \\ s & t & -u \\ v & -w & x \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -e & h & k \\ f & i & -l \\ g & j & m \end{pmatrix}$

Find (a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{B} - \mathbf{A}$

Find the unknown in each of the following:

6. (a) $\begin{pmatrix} x + 4 & y + 6 \\ 12 & 3 + z \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 12 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} a & b & c \end{pmatrix} + \begin{pmatrix} 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & p & 2 \\ q & 0 & r \\ -1 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -2 \\ 0 & s + p & 1 \\ r & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 4 \\ 2 & 5 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} x & 2 \\ 1 & y \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} a & b & c \\ 14 & d & 7 \\ e & 14 & 2 \end{pmatrix} - \begin{pmatrix} -9 & 5 & 2 \\ 7 & -1 & 7 \\ 10 & 8 & 13 \end{pmatrix} = \begin{pmatrix} -3 & 4 & 8 \\ 7 & 2 & 0 \\ 30 & 6 & f \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} p & q \\ r & -s \\ -1 & u \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -4 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} x + y & 1 \\ 1 & x - y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

(h) $\begin{pmatrix} x + y + z \\ y + z \\ 2z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & x^2+4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -4x \\ 2 & 3 \end{pmatrix}$

(j) $\begin{pmatrix} x^2+5 & 0 \\ -1 & y \end{pmatrix} = \begin{pmatrix} 6x & 0 \\ -1 & 4x \end{pmatrix}$

7. Find the matrix x such that x + $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

8. When the rows and columns of a matrix **A** are interchanged such that the first row becomes the first column, second row becomes second column, and so on, the resulting matrix is called the transpose of A written $\mathbf{A}^T$.

For example, if $\mathbf{A} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$ then $\mathbf{A}^T = \begin{pmatrix} a & e & i \\ b & f & j \\ c & g & k \\ d & h & l \end{pmatrix}$

Write down the transpose of each of the following matrices:

(a) $\begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}$ (b) $\mathbf{Q} = \begin{pmatrix} 1 & -2 \\ 4 & 0 \\ 3 & 1 \end{pmatrix}$ (c) $\mathbf{R} = \begin{pmatrix} 1 & -3 & 7 \\ -4 & 9 & 2 \end{pmatrix}$

(d) $\mathbf{S} = \begin{pmatrix} 4 & 7 & 9 \\ 13 & 15 & -2 \\ -1 & 17 & 3 \end{pmatrix}$ (e) $\mathbf{T} = \begin{pmatrix} 3 & -4 & 0 & 8 \\ -1 & 7 & 3 & -6 \\ 6 & 2 & 5 & -1 \end{pmatrix}$ (f) $\mathbf{U} = \begin{pmatrix} 3 & 2 & -5 & 1 & 8 \\ 9 & 3 & 4 & 0 & 2 \\ 16 & 5 & 1 & -15 & 7 \\ 0 & 3 & -8 & 12 & -1 \end{pmatrix}$

9. The number of vehicles passing through a road on Day One were; 10 buses, 6 lorries and 15 cars. On Day Two there were 4 lorries, 20 cars and 7 buses and on Day Three 9 cars, 3 buses and 1 lorry. Tabulate this information in a matrix form and find the total in each category for the three days.

10. The premier league results for four top teams during the first leg of a certain year were:

	Won	*Drawn*	*Lost*	*Points*
Tusker	8	4	6	28
Mathare	5	7	6	22
Ulinzi	6	2	8	20
Gor Mahia	9	6	3	33

In the second leg the results were:

	Won	*Drawn*	*Lost*	*Points*
Mathare	7	2	9	23
Gor Mahia	10	5	3	35
Tusker	11	5	2	38
Ulinzi	8	6	2	30

What were the final results for the teams at the end of the year?

11. The element i^{th} row and j^{th} column of a matric **A** denoted by a_{ij}

    For instance if $\mathbf{A} = \begin{pmatrix} 5 & 6 & 4 \\ 3 & 2 & 0 \\ 2 & 4 & 3 \\ 8 & 5 & 2 \end{pmatrix}$ the element in the second row and third column, a_{23}, is 0. Similarly $a_{41} = 8$. State the values of a_{12}, a_{21}, a_{43}, a_{42}, a_{32}.
12. The element in the i^{th} row and j^{th} column in a matrix **B** is given by $b_{ij} = j - i$. Copy and complete the matrix **B**.

    $$\mathbf{B} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & - & - & - & - \\ -2 & - & - & - & - \\ -3 & - & - & - & - \\ -4 & - & - & - & - \end{pmatrix}$$

8.4: Matrix Multiplication

Consider the matrix **A**. 3A means 3 times **A**, which is equal to; **A** + **A** + **A**.

For example, given $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix}$ then;

$$\begin{aligned} 3\mathbf{A} &= \mathbf{A} + \mathbf{A} + \mathbf{A} \\ &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0+0+0 & 1+1+1 & 2+2+2 \\ 1+1+1 & 3+3+3 & 1+1+1 \\ 2+2+2 & 1+1+1 & 4+4+4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 & 6 \\ 3 & 9 & 3 \\ 6 & 3 & 12 \end{pmatrix} \end{aligned}$$

We can also get the final matrix by multiplying each element of **A** by 3.

$$\begin{aligned} \text{Thus, } 3\mathbf{A} &= 3\begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 & 6 \\ 3 & 9 & 3 \\ 6 & 3 & 12 \end{pmatrix} \end{aligned}$$

In general, if $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and k is a scalar, then;

$$\begin{aligned} k\mathbf{A} &= k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \\ &= \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix} \end{aligned}$$

Example 3

If $\mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, find: (a) 5**B** (b) -3**B** (c) $\frac{1}{2}$**B** (d) 0.1**B**

Solution

(a) $5\mathbf{B} = 5\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 5\times 2 & 5\times 3 & 5\times 1 \\ 5\times 1 & 5\times 0 & 5\times -1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 15 & 5 \\ 5 & 0 & -5 \end{pmatrix}$$

(b) $-3\mathbf{B} = -3\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -3\times 2 & -3\times 3 & -3\times 1 \\ -3\times 1 & -3\times 0 & -3\times -1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -9 & -3 \\ -3 & 0 & 3 \end{pmatrix}$$

(c) $\frac{1}{2}\mathbf{B} = \frac{1}{2}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2}\times 2 & \frac{1}{2}\times 3 & \frac{1}{2}\times 1 \\ \frac{1}{2}\times 1 & \frac{1}{2}\times 0 & \frac{1}{2}\times -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1.5 & 0.5 \\ 0.5 & 0 & -1.5 \end{pmatrix}$$

(d) $0.1\mathbf{B} = 0.1\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 0.1\times 2 & 0.1\times 3 & 0.1\times 1 \\ 0.1\times 1 & 0.1\times 0 & 0.1\times -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 & 0.3 & 0.1 \\ 0.1 & 0 & -0.1 \end{pmatrix}$$

Consider the premier league results for selected football teams within a given period in a certain year.

Team	*Won*	*Drawn*	*Lost*
Chemelil	19	11	6
Nzoia	19	9	8
Coast Stars	14	8	12

For every win, a team scores 3 points, for every draw 1 point and for a loss 0 point. Find the total points for each team.

Total points for Chemelil = $19\times 3 + 11\times 1 + 6\times 0 = 68$
Total points for Nzoia = $19\times 3 + 9\times 1 + 8\times 0 = 66$
Total points for Coast Stars = $14\times 3 + 8\times 1 + 12\times 0 = 50$

To get the total points for Chemilil, the information can be arranged in a matrix form as follows;

$$\begin{pmatrix} 19 & 11 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = (19\times 3 + 11\times 1 + 6\times 0)$$

$$= (68)$$

The matrix (68) is the product of the two matrices. Similarly, for Nzoia and Coast Stars we get;

$$\begin{pmatrix} 19 & 9 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = (19\times 3 + 9\times 1 + 8\times 0)$$

$$= (66)$$

$$(14 \quad 8 \quad 12)\begin{pmatrix}3\\1\\0\end{pmatrix} = (14 \times 3 + 8 \times 1 + 12 \times 0)$$

$$= (50)$$

Since we are multiplying by the same matrix $\begin{pmatrix}3\\1\\0\end{pmatrix}$ in each case, the three results can be combined and written as:

$$\begin{pmatrix}19 & 11 & 6\\19 & 9 & 8\\14 & 8 & 12\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}19 \times 3 + 11 \times 1 + 6 \times 0\\19 \times 3 + 9 \times 1 + 8 \times 0\\14 \times 3 + 8 \times 1 + 12 \times 0\end{pmatrix}$$

$$= \begin{pmatrix}68\\66\\50\end{pmatrix}$$

The matrix $\begin{pmatrix}68\\66\\50\end{pmatrix}$ is the product of the two matrices.

Work out the following:

(i) $(6 \quad 1)\begin{pmatrix}3\\2\end{pmatrix}$

(ii) $(1 \quad 4)\begin{pmatrix}10\\-3\end{pmatrix}$

(iii) $(7 \quad 0 \quad -2)\begin{pmatrix}1\\-3\\2\end{pmatrix}$

(iv) $(-2 \quad -4 \quad 3)\begin{pmatrix}1\\-4\\6\end{pmatrix}$

(v) $\begin{pmatrix}-2 & 4\\3 & -6\end{pmatrix}\begin{pmatrix}-2\\2\end{pmatrix}$

(vi) $\begin{pmatrix}7 & 4\\1 & -2\end{pmatrix}\begin{pmatrix}2\\-3\end{pmatrix}$

(vii) $\begin{pmatrix}-4 & 6 & 2\\3 & -3 & 4\\-5 & 6 & 8\end{pmatrix}\begin{pmatrix}-1\\0\\2\end{pmatrix}$

(viii) $\begin{pmatrix}1 & -3 & 3\\2 & 1 & 0\\3 & 2 & -4\end{pmatrix}\begin{pmatrix}0\\2\\-1\end{pmatrix}$

Example 4

A kiosk owner wanted to buy one *debe* of potatoes, three bunches of bananas and two baskets of onions. He went to Uhuru Market and found the prices as sh. 280 for a *debe* of potatoes; sh. 50 for a bunch of bananas and sh. 100 for a basket of onions. At Harambee Market the corresponding prices were sh. 300, sh. 48 and sh. 80.

(a) Express the kiosk owner's requirements as a row matrix.

(b) Express the prices in each market as a column matrix.

(c) Use the matrices in (a) and (b) to find the total cost in each market.

Solution

(a) Requirements in matrix form is $(1 \quad 3 \quad 2)$

(b) Price matrix for Uhuru Market is $\begin{pmatrix}280\\50\\100\end{pmatrix}$

Price matrix for Harambee Market is $\begin{pmatrix} 300 \\ 48 \\ 80 \end{pmatrix}$

(c) Total cost in shillings at Uhuru Market is;

$$(1 \quad 3 \quad 2)\begin{pmatrix} 280 \\ 50 \\ 100 \end{pmatrix} = (1 \times 280 + 3 \times 50 + 2 \times 100) = (630)$$

Total cost in shillings at Harambee market is;

$$(1 \quad 3 \quad 2)\begin{pmatrix} 300 \\ 48 \\ 80 \end{pmatrix} = (1 \times 300 + 3 \times 48 + 2 \times 80) = (604)$$

The two results can be combined as:

$$(1 \quad 3 \quad 2)\begin{pmatrix} 280 & 300 \\ 50 & 48 \\ 100 & 80 \end{pmatrix} = (1 \text{ x } 280 + 3 \text{ x } 50 + 2 \text{ x } 100 \quad 1 \text{ x } 300 + 3 \text{ x } 48 + 2 \text{ x } 80)$$

$$= (630 \quad 604)$$

Note that in the 3×2 price matrix on the left side, the first column is for Uhuru and the second column for Harambee Market. Similarly, in the product matrix (630 604) on the right hand side, the first column is for Uhuru and the second column for Harambee.

Work out the following products:

(i) $(1 \quad 2)\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $(1 \quad 4)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(iii) $(2 \quad 3 \quad 0)\begin{pmatrix} 1 & 3 & 0 \\ 4 & 1 & 2 \\ 2 & 1 & 5 \end{pmatrix}$

(iv) $(2 \quad 0 \quad 1 \quad 5)\begin{pmatrix} 0 & 6 \\ 2 & 3 \\ 1 & 2 \\ 5 & 4 \end{pmatrix}$

(v) $(0 \quad 5 \quad 3)\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & 6 \end{pmatrix}$

(vi) $(2 \quad 7 \quad 3)\begin{pmatrix} 1 & 0 & 6 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & 5 & 0 & 5 \end{pmatrix}$

Consider the matrices $\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 0 \\ 6 & 7 & 2 \end{pmatrix}$

Let us find the matrix product **AB**.
To get the first row of the product matrix, we multiply the first row of **A** by the matrix **B**.

$$(2 \quad 1 \quad 3)\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 0 \\ 6 & 7 & 2 \end{pmatrix}$$
$$= (2 \times 3 + 1 \times 4 + 3 \times 6 \quad 2 \times 2 + 1 \times 1 + 3 \times 7 \quad 2 \times 5 + 1 \times 0 + 3 \times 2)$$
$$= (28 \quad 26 \quad 16)$$

To get second row of the product matrix, we multiply the second row of **A** by the matrix **B**.

$$\begin{pmatrix}1 & 0 & 4\end{pmatrix}\begin{pmatrix}3 & 2 & 5\\ 4 & 1 & 0\\ 6 & 7 & 2\end{pmatrix}$$

$$= (1\times3+0\times4+4\times6 \quad 1\times2+0\times1+4\times7 \quad 1\times5+0\times0+4\times2)$$

$$= (27 \quad 30 \quad 13)$$

Therefore, the matrix product;

$$\mathbf{AB} = \begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 4\end{pmatrix}\begin{pmatrix}3 & 2 & 5\\ 4 & 1 & 0\\ 6 & 7 & 2\end{pmatrix} = \begin{pmatrix}28 & 26 & 16\\ 27 & 30 & 13\end{pmatrix}$$

Example 5

Given $\mathbf{A} = \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3 & 4 & 3\\ 0 & 2 & 9\\ 4 & 1 & 1\end{pmatrix}$, find the matrix product **AB.**

Solution

$$\mathbf{AB} = \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}3 & 4 & 3\\ 0 & 2 & 9\\ 4 & 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1\times3+2\times0+3\times4 & 1\times4+2\times2+3\times1 & 1\times3+2\times9+3\times1\\ 0\times3+1\times0+2\times4 & 0\times4+1\times2+2\times1 & 0\times3+1\times9+2\times1\end{pmatrix}$$

$$= \begin{pmatrix}15 & 11 & 24\\ 8 & 4 & 11\end{pmatrix}$$

Note:

(i) Each time a row is multiplied by a column.

(ii) In the product matrix, the element in he i^{th} row and j^{th} column is obtained by multiplying the i^{th} row of **A** by the j^{th} column of **B**.

(iii) For the multiplication of **A** by **B** to be possible, the number of columns of **A** must be equal to the number of rows of **B**. The two matrices **A** and **B** are then said to be compatible (conformable).

In general, given a matrix **A** of order m × n and a matrix **B** of order n × p the product matrix **AB** is of order m × p.

Example 6

Given the matrices $\mathbf{A} = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$, find **AB** and **BA**.

Solution

$$\mathbf{AB} = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$$

$$= \begin{pmatrix}2\times1+1\times4 & 2\times0+1\times1\\ 1\times1+3\times4 & 1\times0+3\times1\end{pmatrix}$$

$$= \begin{pmatrix} 2+4 & 0+1 \\ 1+12 & 0+3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 \\ 13 & 3 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+0\times1 & 1\times1+0\times3 \\ 4\times2+1\times1 & 4\times1+1\times3 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 1+0 \\ 8+1 & 4+3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 9 & 7 \end{pmatrix}$$

Note:

AB ≠ BA

It is important to observe the order in which the matrix product is required. If **A** comes before **B**, **B** is said to be pre-multiplied by **A**. If **A** comes after **B**, **B** is said to be post-multiplied by **A**.

Exercise 8.2

1. Given $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}$ find:

   (a) $2P$ (b) $\frac{1}{2}\mathbf{Q}+\mathbf{P}$ (c) $\mathbf{P}+\mathbf{Q}+\mathbf{R}$

   (d) $3\mathbf{P}-2(\mathbf{Q}+\mathbf{R})$ (e) $2\mathbf{Q}-3\mathbf{P}+\mathbf{R}$ (f) $4\mathbf{R}+\mathbf{P}-2\mathbf{Q}$

2. Find the product of the following matrices, where possible:

   (a) $(1 \quad 3)\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}(3 \quad 1)$ (c) $(-2 \quad 4)\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

   (d) $(2 \quad 0 \quad 1)\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ (e) $(1 \quad 3 \quad 2)\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$ (f) $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \\ 9 \end{pmatrix}$

   (g) $(3 \quad 4)(2 \quad 4)$ (h) $\begin{pmatrix} 2 & -1 & 3 \\ 7 & 4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -4 \end{pmatrix}$

   (i) $(1 \quad 0 \quad 3)\begin{pmatrix} 2 & 0 & 2 \\ -3 & 1 & 4 \\ 4 & 2 & -7 \end{pmatrix}$ (j) $\begin{pmatrix} 5 & 4 \\ -1 & 4 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 0 & -5 \end{pmatrix}$

3. Given $\mathbf{L} = \begin{pmatrix} -4 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & 3 \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} 6 & 3 & 0 \\ 0 & 1 & -2 \\ -3 & 3 & 1 \end{pmatrix}$, find $\frac{1}{3}\mathbf{M} - \frac{1}{2}\mathbf{L}$.

4. Work out the following:

   (a) $(1 \quad 0)\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} -1 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} -7 & 4 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} 2 & -9 \\ 1 & 7 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 5 & 4 \\ -8 & -3 & 2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ -1 & 5 \\ 3 & -2 \end{pmatrix}$ (e) $(1 \quad 2 \quad 9)\begin{pmatrix} 3 & 4 & 7 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 5 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 2 & 4 & 1 \\ 2 & 9 & 6 \end{pmatrix}\begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} 0 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$ (i) $\begin{pmatrix} 4 & -3 \\ 2 & 4 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 & -4 \\ 1 & 0 & 0 \end{pmatrix}$

5. Copy and complete table 8.1 for the matrices **A** and **B**.

Table 8.1

Order of A	*Order of B*	*Order of AB*
2 × 2	2 × 1	
2 × 3	3 × 1	
2 × 3	3 × 2	
2 × 4	4 × 1	
3 × 2	2 × 4	3 × 4
3 × 3	3 × 1	
3 × 4	4 × 2	
m × n	n × p	

6. Work out the following:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ (c) $(4 \quad 5)\begin{pmatrix} 3 & 5 & -1 \\ 4 & 6 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} -2 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ (e) $(5 \quad 1)\begin{pmatrix} -2 & 3 \\ 6 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 4 & 7 \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 4 & -1 \\ -2 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 8 & 1 \\ 2 & 0 \end{pmatrix}$ (i) $(1 \quad 2 \quad 3)\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 5 \\ 1 & 3 & 2 \end{pmatrix}$

(j) $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 5 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 0 \\ 3 & 2 & 4 \end{pmatrix}$

7. Find the unknown in each of the following:

(a) $(1 \quad 3)\begin{pmatrix} 2x & 9 \\ 1 & 3y \end{pmatrix} = (7 \quad 9)$ (b) $(x \quad y)\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} = (4 \quad 1)$

(c) $(a \quad 22)\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} = (9 \quad 14)$ (d) $\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

(e) $(2x \quad x)\begin{pmatrix} x \\ 4 \end{pmatrix} = (6)$ (f) $(x \quad y)\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = (7 \quad 17)$

(g) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & -7 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

8. If $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}$, evaluate:
   (a) $(\mathbf{A} + \mathbf{B})^2$ (b) $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$
   Is $(\mathbf{A} + \mathbf{B})^2$ equal to $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$?
9. If $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 2 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$, evaluate:
   (a) (i) $\mathbf{A}(\mathbf{B} + \mathbf{C})$ (ii) $\mathbf{AB} + \mathbf{AC}$
   (b) (i) $(\mathbf{A} + \mathbf{B})\mathbf{C}$ (ii) $\mathbf{AC} + \mathbf{BC}$
   (c) (i) $(\mathbf{AB})\mathbf{C}$ (ii) $\mathbf{A}(\mathbf{BC})$
   What do you notice in (i) and (ii) in each case?
10. Two cinema theatres A and B carry 700 people each. Each of them carries 300 people upstairs and 400 downstairs. Theatre A charges sh. 150 for upstairs and sh. 100 for downstairs. Theatre B charges sh. 140 upstairs and sh. 90 downstairs. Using matrix method, calculate the total collection for each theatre during a show when all seats are booked.
11. In a supermarket X, the price of $\frac{1}{2}$ kilogram detergent is sh. 50, 500 grams packet of tea costs sh. 60 and 1 kilogram sugar sh. 45. In supermarket Y, the prices are sh. 55, sh. 62 and sh. 49 respectively.

    Mrs. Kamau bought $2\frac{1}{2}$ kg of detergent, four 500 grams packets of tea and 3 kg sugar but in supermarket X. Mrs Mutua bought the same amount of the items but in supermarket Y. Using matrices, find the total amount spent by each lady.
12. Bookings for train seats between Nairobi and two stations K and M were equal. There were 50 in first class, 90 in second class and 260 in third class in each train. For station K, the fares were first class sh. 450, second class sh. 156 and third class sh. 56. For station M, the fares were first class sh. 490, second class sh. 210 and third class sh. 70. Using matrices, find the total collection.
13. In a Form 3A, three tests are done in first term, four in second term and two in third term. The first five students A, B, C, D and E had the following average marks:

Student	*Term 1*	*Term 2*	*Term 3*
A	69	58	43
B	80	36	62
C	72	54	48
D	65	46	73
E	47	67	70

Arrange this information in matrix form and find the best student that year in the class.

14. A school buys 20 bags of rice, 30 bags of maize and 20 bags of beans each school term. If the prices per bag of rice, maize and beans are sh. 1400, sh. 1200 and sh. 2000 respectively, form matrices using the information. Hence calculate the total cost of the foodstuff to the school.
15. Joyce bought 5 oranges, 4 apples and 10 bananas. Margaret bought 6 oranges, 2 apples and 15 bananas. If the prices for each orange, apple and banana were sh. 5.00, Sh. 20.00 and sh. 3.00 respectively, form matrices from the information and find the total expenditure of each person.
16. In English soccer results, a win scores 3 points, a draw 1 point and a loss no points. By forming matrices, find the best team out of the following:

Table 8.2

Team	*Wins*	*Draws*	*Losses*
Manchester United	15	1	3
Arsenal	13	6	0
Chelsea	13	3	3
Charlton	8	6	5
Fulham	8	4	7
Liverpool	7	5	7

17. An investor develops a site by building 5 blocks of flats, 20 maisonettes and 10 bungalows. Write this information as a 1 × 3 matrix P. One block of flats requires 5 000 units of material, a maisonette 2 000 units and a bungalow 2 800 units. The number of hours of labour required for each flat, maisonette and bungalow are 900 hrs, 400 hrs and 360 hrs respectively. Write down a 3 × 2 inputs matrix **Q**.

    What does the matrix product **PQ** represent? If the labour costs average sh. 50 per hour and materials cost sh. 100 per unit, write down a cost matrix **R** and hence, find the product **PQR**. What does this product represent?

8.5: Identity Matrix

Consider matrices **A** and **B** of order 2 × 2 below:

$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Multiply each matrix by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(i) What do you notice?

(ii) If you reverse the order of multiplication of the matrices, what do you notice?

Now, multiply each of the matrices $\mathbf{P} = \begin{pmatrix} 4 & 2 & 0 \\ 1 & 3 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

What do you notice?

You should have noticed that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ behaves like the number 1 in the multiplication of numbers,

Thus, $5 \times 1 = 1 \times 5 = 5$

$a \times 1 = 1 \times a = a$

The matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ behaves similarly.

A square matrix which has ones in the **main diagonal** and zeros elsewhere is called a **unit** or an **identity** matrix. The main diagonal is the one running from top left to the bottom right. It is also called **principal** or **leading diagonal.** The letter I is used to denote the identity matrix.

The identity matrix of order 2, denoted by $\mathbf{I}_2$, is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The identity matrix of order 3, $\mathbf{I}_3$, is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Write down $\mathbf{I}_4$ and $\mathbf{I}_5$.

In general, if **M** is a matrix and **I** is an identity matrix of the same order, then **MI** = **IM** = **M**.

8.6: Determinant of a 2 × 2 Matrix

For a square matrix of order two, if we subtract the product of the elements of the other diagonal from the product of the elements of the main diagonal, the number obtained is called the **determinant** of the matrix.

If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$, the determinant of **A** is; $(3 \times 5) - (2 \times 4) = 7$.

It is written in short as det **A** = 7.

Generally, if $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then, det **B** = ad − bc.

State the determinant of each of the following matrices:

(i) $\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 2 \\ 7 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & -5 \\ 3 & 6 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & 3 \\ -3 & 7 \end{pmatrix}$ (v) $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ (vi) $\begin{pmatrix} -2 & 2 \\ 6 & -9 \end{pmatrix}$

8.7: Inverse of a 2 × 2 Matrix

Consider Matrices **P** and **Q** below;

$\mathbf{P} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$

The products **PQ** and **QP** are;

$\mathbf{PQ} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ $\mathbf{QP} = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

You realise that in both cases, the product is an identity matrix.
If **PQ = QP = I**, then **Q** is the inverse of **P** and vice versa.
The inverse of **A** is written as $\mathbf{A}^{-1}$
Therefore $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Example 7

Show that $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ is the Inverse of $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

Solution

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 3 + 1 \times -5 & 2 \times -1 + 1 \times 2 \\ 5 \times 3 + 3 \times -5 & 5 \times -1 + 3 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 2 + -1 \times 5 & 3 \times 1 + -1 \times 3 \\ -5 \times 2 + 2 \times 5 & -5 \times 1 + 2 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

AB = BA = I. Hence, A is the inverse of **B**.
Verify that the following pairs of matrices are inverses of each other:

(i) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -5 \\ -1 & 7 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$ and $\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$ (v) $\begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$ (vi) $\begin{pmatrix} 5 & 7 \\ -3 & -4 \end{pmatrix}$ and $\begin{pmatrix} -4 & -7 \\ 3 & 5 \end{pmatrix}$

For each pair of matrices in the exercise above, compare the elements in the main diagonals. What do you notice? Compare also the elements in the other diagonals.

You should have noticed that:

(a) the elements in the main diagonal are interchanged.

(b) the signs of the elements in the other diagonal are reversed.

Find the inverse of each of the following matrices. Check your answer by multiplication:

(i) $\begin{pmatrix} -2 & -1 \\ 11 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & 17 \\ 1 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & -9 \\ -1 & 5 \end{pmatrix}$ (v) $\begin{pmatrix} 4 & 5 \\ -5 & -6 \end{pmatrix}$ (vi) $\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix}$

Notice that in all the above matrices, the determinant of each matrix is equal to 1.

Can we find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 4 & 2 \end{pmatrix}$ whose determinant is not 1 using the same procedure? Applying observations (a) and (b) above on matrix **A**, we obtain the matrix $\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$.

But $\begin{pmatrix} 8 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 4\mathbf{I}$

and $\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 8 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 4\mathbf{I}$

Therefore, $\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$ is not the inverse of $\begin{pmatrix} 8 & 3 \\ 4 & 2 \end{pmatrix}$

We have obtained $4 \times \mathbf{I}$

Note that 4 is the determinant of $\begin{pmatrix} 8 & 3 \\ 4 & 2 \end{pmatrix}$

To obtain the inverse of **A** we divide $\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$ by 4.

$$\text{Therefore } A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ -1 & 2 \end{pmatrix}$$

The procedure for getting the inverse of a 2 × 2 matrix is;

(i) Find the determinant of the matrix. If it is zero, then there is no inverse.

(ii) If it is non zero, then:

- interchange the elements in the main diagonal.
- reverse the signs of the elements in the other diagonal.
- divide the matrix obtained by the determinant of the given matrix.

Example 8

Find the inverse of $\mathbf{A} = \begin{pmatrix} 4 & 8 \\ 3 & 5 \end{pmatrix}$

Solution

$$\begin{aligned} \text{Det } \mathbf{A} &= (4 \times 5) - (8 \times 3) \\ &= 20 - 24 \\ &= -4 \end{aligned}$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 5 & -8 \\ -3 & 4 \end{pmatrix}$$

$$= \frac{-1}{4}\begin{pmatrix} 5 & -8 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{4} & 2 \\ \frac{3}{4} & -1 \end{pmatrix}$$

Alternatively;

Let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & 8 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$4a + 3b = 1$ ... ...(1)

$8a + 5b = 0$........(2)

$\Rightarrow \quad 8a + 6b = 2$

$\underline{8a + 5b = 0}$

$b = 2$

Substitution in (2) gives;

$4a + b = 1$

$4a = -5$

$a = -\frac{5}{4}$

Also;

$4c + 3d = 0$ ....(3) $\quad \Rightarrow 8c + 6d = 0$

$8c + 5d = 1$ ....(4) $\quad \underline{8c + 5d = 1}$

$d = -1$

Substituting in (3);

$4c - 3 = 0$

$c = \frac{3}{4}$

$\therefore \; A^{-1} = \begin{pmatrix} -\frac{5}{4} & 2 \\ \frac{3}{4} & -1 \end{pmatrix}$

In general, if $x = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$\det x = ps - qr$, and,

$x^{-1} = \frac{1}{\det x}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$

$= \frac{1}{ps - qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$, provided (ps – qr).

If the determinant is zero, such a matrix has no inverse and is called a **singular matrix**.

Determine the singular matrices among the following:

(i) $\begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ (v) $\begin{pmatrix} 5 & 5 \\ 3 & 2 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Exercise 8.3

1. Find the determinant of each of the following matrices:

   (a) $\begin{pmatrix} 3 & -2 \\ 7 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 5 \\ 5 & -8 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & -4 \\ -2 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 4 \\ 6 & 12 \end{pmatrix}$

   (e) $\begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$ (g) $\begin{pmatrix} 5 & 1 \\ 6 & 4 \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

   (i) $\begin{pmatrix} 2 & -10 \\ 1 & 0 \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$ (k) $\begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{3} \end{pmatrix}$ (l) $\begin{pmatrix} x & 2 \\ -y & 3 \end{pmatrix}$

   (m) $\begin{pmatrix} 20 & 16 \\ 3 & 2 \end{pmatrix}$ (n) $\begin{pmatrix} -4 & -2 \\ 3 & -4 \end{pmatrix}$ (p) $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

2. Find where possible the inverse of each of the following matrices:

   (a) $\begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 13 \\ -2 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} 8 & -7 \\ 4 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 6 & 2 \\ 4 & -8 \end{pmatrix}$

   (e) $\begin{pmatrix} 3 & 5 \\ 9 & 15 \end{pmatrix}$ (f) $\begin{pmatrix} 5 & 13 \\ 3 & 8 \end{pmatrix}$ (g) $\begin{pmatrix} 5 & 10 \\ 5 & 9 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

   (i) $\begin{pmatrix} 8 & -4 \\ -3 & 2 \end{pmatrix}$ (j) $\begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (k) $\begin{pmatrix} 13 & 0 \\ 7 & -1 \end{pmatrix}$ (l) $\begin{pmatrix} 4 & -2 \\ -12 & -6 \end{pmatrix}$

   (m) $\begin{pmatrix} 2 & -2 \\ 3 & 1 \end{pmatrix}$ (n) $\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$ (p) $\begin{pmatrix} 8 & 2 \\ 4 & \frac{1}{2} \end{pmatrix}$

3. Solve for the unknowns in each of the following:

   (a) $\begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. Find the value of the unknown if each of the following matrices is singular:

   (a) $\begin{pmatrix} 4 & 3 \\ 5 & c \end{pmatrix}$ (b) $\begin{pmatrix} 4 & y^2 \\ 4 & y \end{pmatrix}$ (c) $\begin{pmatrix} 2x - 1 & 1 \\ x^2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} x + 7 & 4 \\ -3 & x \end{pmatrix}$

5. Given that $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix}$, find:

   (a) $\mathbf{AB}^{-1}$

   (b) $(\mathbf{A} + \mathbf{B})^{-1}$

8.8: Solutions of Simultaneous Linear Equations using Matrices

Consider the simultaneous equations:

$3x + y = 7$

$5x + 2y = 12$

Using elimination, substitution or graphical method, we obtain $x = 2$ and $y = 1$. The matrix method can also be used to solve simultaneous equations. The above equations are in this case written as: $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$

The Matrix $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is called the **co-efficients matrix** of the simultaneous equations, while $\begin{pmatrix} 7 \\ 12 \end{pmatrix}$ is called the **constants matrix**.

Pre-multiplying both sides of the matrix equation by the inverse of the co-efficients matrix.

$$\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore x = 2, y = 1$

Example 9

Use the matrix method to solve the following pair of simultaneous equations:

$3a + 2b = 12$

$4a - b = 5$

Solution

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

The inverse of the co-efficients matrix is $-\frac{1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}$

Pre-multiplying both sides of the matrix equation by the inverse;

$$-\frac{1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$-\frac{1}{11}\begin{pmatrix} -11 & 0 \\ 0 & -11 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -22 \\ -33 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore, $a = 2$ and $b = 3$.

Use the matrix method to solve, where possible, the following pairs of simultaneous equations:

(i) $3y - 2x = 3$
$3y + x = 4$

(ii) $y + 2x = 4$
$2y + x = 5$

(iii) $2c + 3d = -2$
$3c + d = 4$

(iv) $e + 4f = 4$
$3e - 2f = 5$

(v) $-2y + 3x = 11$
$x + y = 2$

(vi) $2x + y = 7$
$4x + 3y = 17$

(vii) $2p + 3q = 15$
$7q + 5p = -13$

(viii) $5u + 4v = 9$
$2v + 7u = 2$

(ix) $3x + 4y = -5$
$2y - x = \frac{9}{2}$

(x) $3x + y = 8$
$-y + 2x = -3$

(xi) $x + 3y = 5$
$2x + 6y = 7$

(xii) $5x + 2y = 1$
$10x + 4y = 2$

Suppose we wish to solve the simultaneous equations;
$2x + 4y = 6$
$3x + 6y = 5$
written in the matrix form the equations are.

$$\begin{pmatrix}2 & 4\\3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6\\5\end{pmatrix}$$

The determinant of the co-efficients matrix is zero.
Therefore, the matrix has no inverse.
We need to re-examine the equations. The equations may be re-written as:
$y = -\frac{1}{2}x + \frac{3}{2}$ and $y = -\frac{1}{2}x + \frac{5}{6}$
We notice that the gradients are the same, but y-intercepts different.
Hence the two lines are parallel, as shown in figure 8.1.

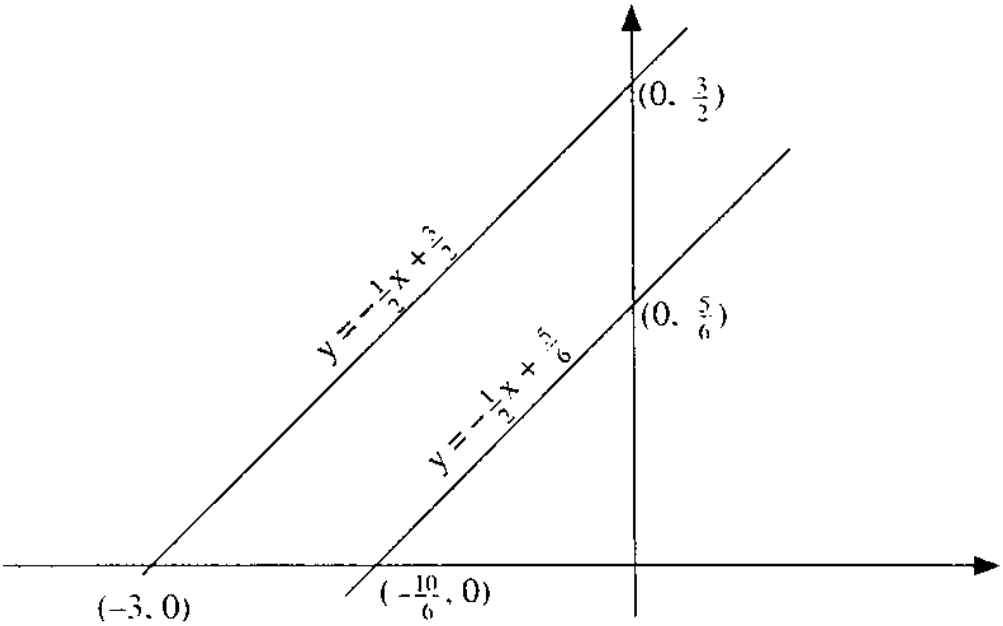


Fig. 8.2

Since the lines do not intersect, there is no solution.

Consider the simuaneous equations.

$x - 2y = 2$

$2x - 4y = 4$

Employing the matrix method;

$$\begin{pmatrix}1 & -2\\ 2 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2\\ 4\end{pmatrix}$$

Here again, the coefficients matrix has no inverse. When we re-examine the equations, we notice that one equation is a multiple of the other. The two equations therefore represent the same line. Every point of this line is a solution to the two equations. Thus, there are infinitely many solutions.

Generally, if the determinant of the coefficient is zero, either the linear simultaneous equations have no solution (parallel lines) or they have infinitely many solutions (coincidental lines).

Exercise 8.4

Use the matrix method to solve the following pairs of simultaneous equations:

1. (a) $\begin{array}{l} x + y = 5 \\ 3x - 2y = 0 \end{array}$ (b) $\begin{array}{l} 4p + q = 6 \\ 2p - q = -3 \end{array}$ (c) $\begin{array}{l} 3m + n = 8 \\ -n + 4m = 1 \end{array}$
   (d) $\begin{array}{l} 3t - r = 5 \\ 5t - 2r = 9 \end{array}$ (e) $\begin{array}{l} 4x - 3y = 1 \\ 2x + 6y = 3 \end{array}$
2. (a) $\begin{array}{l} x - 2y = 4 \\ 2x + y = 3 \end{array}$ (b) $\begin{array}{l} x + y = 0 \\ x - y = 4 \end{array}$ (c) $\begin{array}{l} a - 2b = 2 \\ 2a - 3b = 1 \end{array}$
   (d) $\begin{array}{l} 2c - d = -3 \\ -2d + c = 0 \end{array}$ (e) $\begin{array}{l} v - 3u = -2 \\ 3u - 2v = -2 \end{array}$
3. (a) $\begin{array}{l} x + 2y = 6 \\ 15x - 3y = 3 \end{array}$ (b) $\begin{array}{l} 4y + 2x = 12 \\ x + 3y = 8 \end{array}$ (c) $\begin{array}{l} 5w - 2x = -3 \\ -w + 3x = 11 \end{array}$
   (d) $\begin{array}{l} p - q = -5 \\ -2q + 3p = -14 \end{array}$ (e) $\begin{array}{l} 0.8x - 0.5y = 0.85 \\ -2.4x + 1.7y = 3.05 \end{array}$
4. (a) $\begin{array}{l} 2a + b = 10 \\ -a + 2b = 5 \end{array}$ (b) $\begin{array}{l} 2m + 5n = 2 \\ 4m - 3n = 1\frac{2}{5} \end{array}$ (c) $\begin{array}{l} 3g - f = -5 \\ 4f + 2g = 7 \end{array}$
   (d) $\begin{array}{l} 3u - 2v = 0 \\ -u + v = 1 \end{array}$ (e) $\begin{array}{l} \frac{1}{2}u + v = 8 \\ \frac{3}{2}u - \frac{1}{3}v = 4 \end{array}$
5. (a) $\begin{array}{l} 5x + 3y = 7 \\ 2x + y = 5 \end{array}$ (b) $\begin{array}{l} 7p + 5q = 11 \\ 3q + 4p = 2 \end{array}$ (c) $\begin{array}{l} 3x + 2 = y \\ -4 + 2y = 2x \end{array}$
   (d) $\begin{array}{l} \frac{1}{2}c + \frac{1}{3}d = 24 \\ \frac{1}{2}c + \frac{1}{4}d = -6 \end{array}$ (e) $\begin{array}{l} 5t - 3s = -3 \\ 7t = 1 \end{array}$

Chapter Nine

FORMULAE AND VARIATIONS

9.1: Formulae

The area (A) of a triangle whose length is (*l*) and width (w) is given by $A = lw$. The equation $A = lw$ is the formula for finding the area of a rectangle. The formula for finding the area of a circle is $A = \pi r^2$, where A is the area and r the radius of the circle.

A **formula** is a mathematical relationship connecting two or more quantities. It is an equation in which letters represent quantities, for example, $V = lbh$ where V is the volume of a cuboid, *l* the length, b the breath and h the height.

State the formula for finding:

(i) area of a triangle (ii) area of a trapezium (iii) volume of a cuboid

Changing the Subject of a Formula

In the formula $V = \pi r^2h$, V is expressed in terms of r and h. V is said to be the subject of the formula. h or r can be made the subject of the same formula as follows:

(i) $V = \pi r^2h$

Dividing both sides by πr^2:

$$\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$

$$\therefore \quad h = \frac{V}{\pi r^2}$$

(ii) $V = \pi r^2h$

Dividing both sides by πh;

$$\frac{V}{\pi h} = \frac{\pi r^2 h}{\pi h}$$

$$r^2 = \frac{V}{\pi h}$$

Taking square root on both sides;

$$r = \sqrt{\frac{V}{\pi h}}$$

Example 1

Given the formula $v = u + at$:

(a) Make t the subject of the formula.

(b) find t when $v = 15\ mS^{-1}$, $a = 2mS^{-2}$ and $u = 3\ mS^{-1}$.

Solution

(a) $v = u + at$

$v - u = at$

$t = \frac{v - u}{a}$

(b) $t = \frac{v - u}{a}$

$= \frac{15 - 3}{2}$

$= \frac{12}{2}$

$= 6$

Example 2

Given the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$:

(a) make v the subject of the formula.

(b) find v when u = 30 cm and f = 10 cm.

Solution

(a) $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

Multiplying both sides by uvf, the L.C.M. of u, v and f;

$uv = vf + fu$

Subtraction vf on both sides;

$uv - vf = fu$

Factorising the L.H.S;

$v(u - f) = fu$

Dividing both sides by v – f

$v = \frac{fu}{u - f}$

Alternatively;

$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

Subtracting $\frac{1}{u}$ on both sides;

$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$

$\frac{1}{v} = \frac{u - f}{fu}$

Taking the reciprocals on both sides

$v = \frac{fu}{u - f}$

(b) $v = \frac{fu}{u - f}$

$= \frac{10 \times 30}{30 - 10}$

$= \frac{300}{20}$

$= 15$ cm

Example 3

Make d the subject of the formula $G = \sqrt{\left(\frac{d - x}{d - 1}\right)}$

Solution

Squaring both sides;

$G^2 = \frac{d - x}{d - 1}$

Multiplying both sides by (d – 1);

$G^2(d - 1) = d - x$

Expanding the L.H.S;

$dG^2 - G^2 = d - x$

Collecting the terms containing d on the L.H.S.

$dG^2 - d = G^2 - x$

Factorising the L.H.S;

$d(G^2 - 1) = G^2 - x$

Dividing both sides by $G^2 - 1$;

$d = \frac{G^2 - x}{G^2 - 1}$

Exercise 9.1

1. Make the letter in the brackets the subject of the formula:

   (a) $\frac{PRT}{100} = I$ (P) (b) $s = ut + \frac{1}{2}at^2$ (a)

   (c) $T = 2\pi\sqrt{\frac{l}{g}}$ (g) (d) $v^2 - u^2 = 2as$ (s)

   (e) $F = \frac{m_1 m_2}{r^2}$ (r) (f) $V = \frac{4}{3}\pi r^3$ (r)

   (g) $\frac{r_1}{r_2} = \sqrt{\frac{d_2}{d_1}}$ (d_1) (h) $s = \frac{n}{2}[2a + (n - 1)d]$ (d)

   (i) $S = 2\pi r^2 + 2\pi rh$ (h) (j) $A = \pi r^2 + \pi r\sqrt{h^2 + r^2}$ (h)

   (k) $ax^2 + bx + c = 0$ (x) (l) $s = ut + \frac{1}{2}at^2$ (t)

2. Given the formula $T = 2\pi\sqrt{\frac{l}{g}}$;

   (a) make *l* the subject.

   (b) find *l* when T = 20 seconds and $g = 10\ ms^{-2}$

3. (a) Make g the subject of the formula;

   $E = \frac{mv^2}{2} + mgh$

   (b) Find g if E = 1 200 joules, m = 18.5 kg, $v = 10\ ms^{-1}$ and h = 1.5 m.

4. (a) Make P the subject of the formula $\frac{1}{R} = \frac{1}{P} + \frac{1}{Q}$

   (b) Find P given that Q = 6 cm when R = 2.4 cm.

5. (a) Given that $S = \frac{a(1 - r^n)}{1 - r}$ make n the subject.
   (b) Find n when a = 2, r = 0.5 and S = 3.875
6. A law relating pressure (P) and volume (V) is given by $PV^n = c$.
   (a) Make n the subject of the formula.
   (b) Find n when P = 960 mmHg, V = 400 mm^3 and c = 2 120
7. Given that $a = 100b^n$, make n the subject.
8. Make z the subject of the formula $x = \frac{d - dz^2}{fz^2 - g}$
9. Given that $B = \frac{-EN}{\sqrt{N^2 + P}}$, make N the subject of the formula.
10. Make a the subject of the formula $a^2 = (a - b)(a - c)$.
11. Given that $A = \pi(R - r)(R + r)$, make R the subject.

9.2: Variation

The formula $C = \pi d$ is used to find the circumference of a circle given its diameter d. The circumference of a circle thus depends on its diameter. When the diameter changes, the circumference also changes. In this case the circumference (C) and the diameter (d) are called variables. π is fixed and is called a constant. When two or more variables are related, a change in one or more of the variables will result in a change in the other variables. This is variation. There are different types of variations, namely, direct, inverse, partial and joint variations.

Direct Variation

Table 9.1 below shows the cost of various numbers of bottles of milk at sh. 20 per bottle.

Table 9.1

No of bottles of milk	1	2	3	4	5	6
Cost (sh)	20	40	60	80	100	120

From the table, we notice that if the number of bottles doubles, the cost also doubles and if it triples, the cost also triples. The cost of milk is said to be directly proportional to the number of bottles bought. Another way of putting this is that the cost of milk (C) varies directly as the number of bottles bought (N). Mathematically, this is written as $C \propto N$. The symbol '$\propto$' stands for 'varies as' or is proportional to.

Generally, $C \propto N$ means that $\frac{C}{N} = k$ or $C = kN$ where k is a constant.

This constant called the **constant of proportionality**. In table 9.1 is the constant;

$k = \frac{20}{1} = \frac{40}{2} = \frac{60}{3} = 20$

i.e., $\frac{C}{N} = 20$ or $C = 20N$

If a graph of C against N is drawn, it will be a straight line passing through the origin and with gradient 20. Generally, the graph of two linear quantities which are in direct variation is a straight line through the origin. The constant of proportionality is the gradient of the straight line.

Example 4

The table below shows the distance (D) covered by a moving object in a given time (t).

Table 9.2

Time (h)	1	2	3	4	5
Distance (km)	15	30	45	60	75

(a) Draw the graph representing this information.

(b) Use the graph to find the constant of proportionality.

Solution

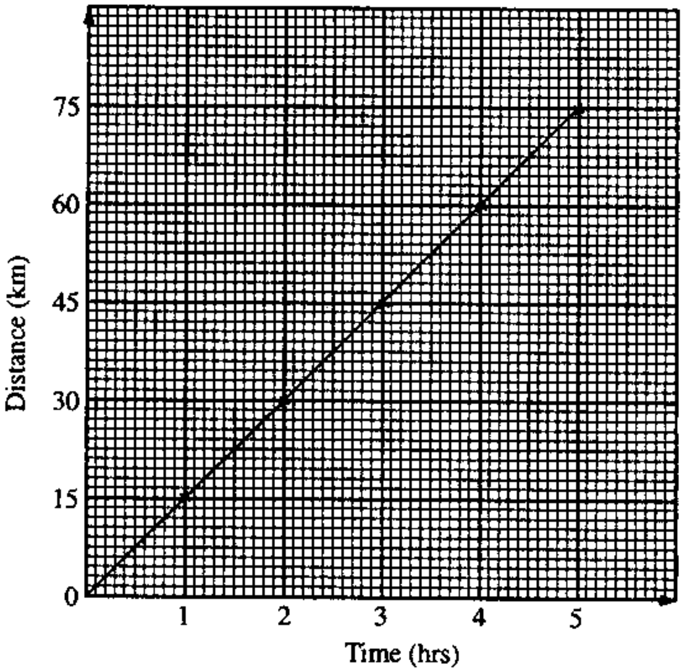


Fig 9.1

$$\text{Gradient} = \frac{60-30}{4-2} = \frac{30}{2} = 15$$

Gradient = $\frac{D}{t}$ = 15

D = 15t is the equation relating D and t and 15 is the constant of proportionality.

Example 5

The length (l) cm of a wire varies directly as the temperature T°C. The length of the wire is 5 cm when the temperature is 65°C. Calculate the length of the wire when the temperature is 69°C.

Solution

$l \propto T$

$\therefore l = kT$

Substituting $l = 5$ when $T = 65$ °C

$5 = k \times 65$

$k = \frac{5}{65} = \frac{1}{13}$

$\therefore l = \frac{1}{13}T$

When $t = 69$

$l = \frac{1}{13} \times 69$

$= 5\frac{4}{13}$ cm

Exercise 9.2

1. Given that a quantity E varies as a quantity M, find:
   (a) the missing values in table 9.3 below.
   (b) the equation connecting E and M.

Table 9.3

M	2	—	5
E	1.2	1.8	—

2. P varies directly as Q. If P = 8 and Q = 48, find:
   (a) the equation connecting P and Q.
   (b) P when Q = 50.
   (c) Q when P = 12.
3. A quantity C varies directly as a quantity D. If C = 22 cm when D = 7 cm, find C when D = 28 cm.
4. Given that y varies directly as the square of x and that x = 2 when y = 3, find y when x = 12.
5. Given that a is directly proportional to b and the difference in values of a when b = 2 and b = 4 is 8. Find the constant of proportionality.

6. If M varies directly as n^2, how is M affected when n is halved?
7. The volume (V) of water in a well varies directly as the depth (d). If the volume is 3 m^3 when the depth is 2 m, determine the volume when the depth is 10 m. If the cross-section of the well is rectangular, find its width given that its length is $2\frac{1}{4}$ m.
8. The volume (V) of an inflated balloon varies as the cube of the diameter (d). The volume is 6.2 cm^3 when its diameter is 2.5 cm. What is the volume of the balloon when its diameter is 3.5 cm?
9. The area (A) of a triangle varies directly as its height (h). The area is 12 cm^2 when the height is 6 cm. What is its Area when its height is 8 cm?

   Find the length of the base of the triangle.
10. The compression (C) cm of a string is directly proportional to the thrust (T) newtons exerted on it. If a thrust of 2 N produces a compression of 0.4 cm, find:
    (a) the compression when the thrust is 5 N
    (b) the thrust when the compression is 0.7 cm.
11. The power (P) kw developed by the engine of a car travelling on a level road varies directly as the cube of the speed (v) kmh^{-1}. If it develops a power of 4000 kw when the speed is 100 Kmh^{-1}, find the power developed by the engine when the speed on level ground is 48 kmh^{-1}.

Inverse Variation

Table 9.4 below shows time taken to cover a distance of 120 km at different speeds.

Table 9.4

Time (t) hrs	1	2	3	4	5	6
Speed (s) km	120	60	40	30	24	20

From the table it can be noticed that:

(i) when the speed is doubled, the time is halved and when tripled, the time is divided by three etc.

(ii) the product of corresponding time and speed is always a constant

$s \times t = 120$

$t = 120 \times \frac{1}{S}$ or $S = 120 \times \frac{1}{t}$

We say that speed is inversely proportional to time, or speed varies inversely as time, i.e., $s \propto \frac{1}{t}$.

In general if y varies inversely as x, then $y = k\frac{1}{x}$, where k is a constant of proportionality. When the speed is plotted against time, the graph obtained is a curve as shown in figure 9.2.

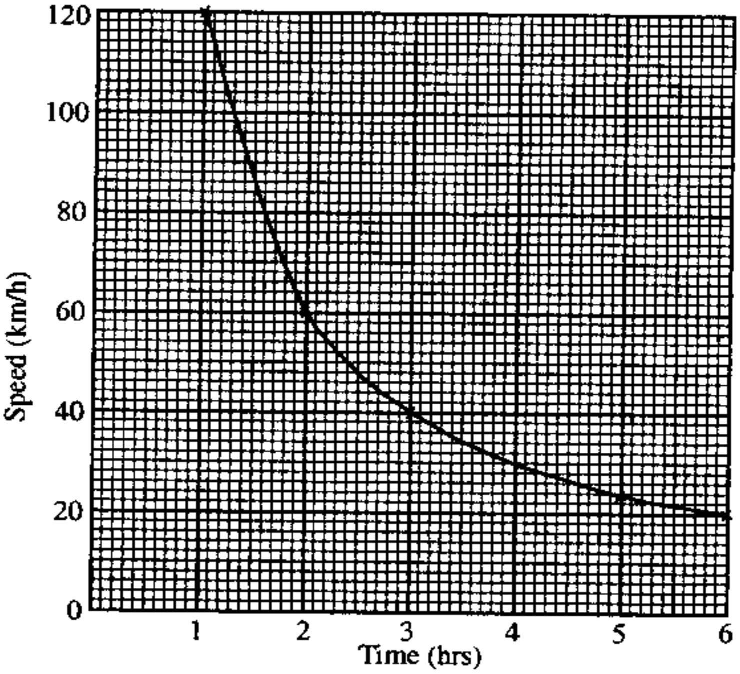


Fig. 9.2

From the graph, the product of any ordered pair on the curve is 120. If instead we plot s against $\frac{1}{t}$ we obtain a straight line graph through the origin, as shown in figure 9.3.

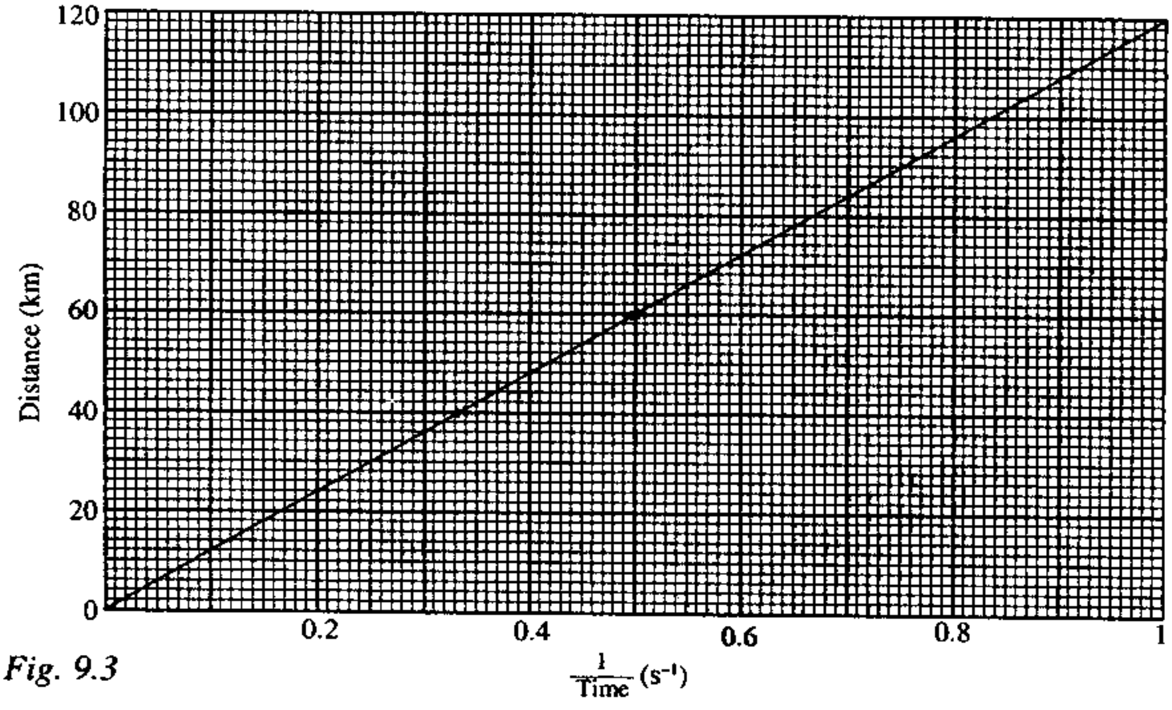


Fig. 9.3

Note that the gradient of the line is 12.

Example 6

It is given that the length, l cm, of a rectangle varies inversely as its width w cm.

(a) If the length is 10 cm when the width is 4 cm, find the length when the width is 5 cm.

(b) Draw the graph of:

(i) l against W

(ii) l against $\frac{1}{W}$

Solution

(a) $l \propto \frac{1}{W}$

Hence, $l = k\frac{1}{W}$

Substituting, $l = 10$, $w = 4$

$10 = k \times \frac{1}{4}$

$k = 40$

Therefore, $l = \frac{40}{W}$

When the width is 5, then;

$l = \frac{40}{5}$

$= 8$ cm

(b) *Table 9.5*

(i)

w	1	2	4	5
l	40	20	10	8

(ii)

w	1	2	4	5
$\frac{1}{W}$	1	0.5	0.25	0.2
l	40	20	10	8

The graphs in figure 9.4 (a) and (b) are drawn using values in tables 9.5 (i) and (ii) respectively.

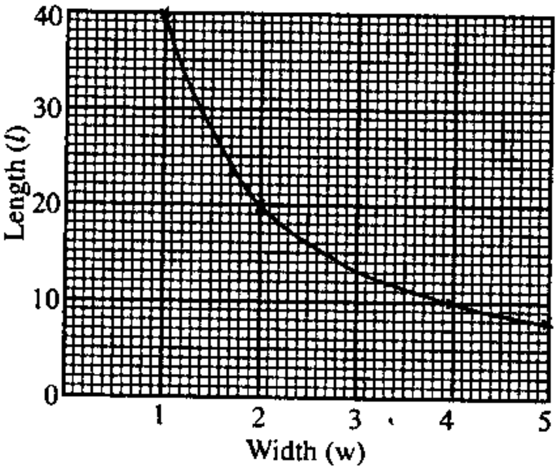


Fig 9.4 (a)

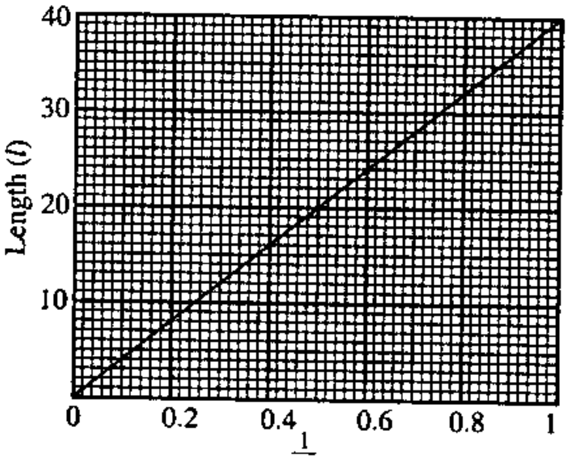


Fig 9.4 (b)

What is the gradient of the line in figure 9.4 (b)?

Exercise 9.3

1. It is known that $P \propto \frac{1}{q}$. If P = 30 when q = 2, find p when $q = \frac{1}{3}$.
2. Given that $y \propto \frac{1}{x}$ and that when y = 6, x = 44, find:
   (a) the equation connecting x and y.
   (b) y when x = 2.
3. A quantity P varies inversely as a quantity Q. Copy and complete the table below.

Table 9.6

Q	15	30	50	75
P	—	5	3	—

Use the table to:
(a) find the constant of proportionality and write an equation relating P to Q.
(b) draw the graph of P against Q.
(c) draw the graph of P against $\frac{1}{Q}$.

4. It is given that p varies inversely as q and that when q = 3, $p = \frac{1}{6}$. Find p when q = 9.
5. Pressure (P) varies inversely as the area (A) it acts upon. Find the equation connecting P and A if P = 60 when A = 4.
6. The values in table 9.7 below are obtained from the relationship $y = \frac{k}{x}$, where k is a constant.

Table 9.7

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
y	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

Use the table to:
(a) find the value of k.
(b) draw the graph of $y = \frac{k}{x}$.

7. Find the constant of proportionality in each of the following:
   (a) $P \propto \frac{1}{V}$, if P = 40 when V = 12.
   (b) $R \propto \frac{1}{r^2}$, if $R = \frac{1}{2}$ when r = 2.

(c) $y \propto \frac{1}{\sqrt[3]{x}}$, if $x = 8$ when $y = 3$.

8. Given that P is inversely proportional to V, copy and complete table 9.8 below

Table 9.8

P	1	3	—	9	12	—	18
V	3	—	0.5	—	0.25	0.2	—

Find the equation connecting P and V.

9. The volume of a certain gas varies inversely as its pressure when the temperature is constant. If 20 litres of the gas support 715 mm of mercury, calculate:
   (a) the volume of the gas when the pressure is 760 mm.
   (b) the pressure of the gas when the volume is 30 litres.
10. The acceleration of a rocket varies inversely as the mass of the rocket if the propelling force is constant. Given that the acceleration of the rocket is 20 ms^{-2} when its mass is 40 tonnes, what is the acceleration when the mass of the rocket is 5 tonnes?
11. The height of a cylinder of constant volume is inversely proportional to its cross-section area. When the area is 612 cm^2, the height is 20 cm. Find the radius of the cylinder when the height is 10 cm.

Partial Variation

The general linear equation $y = mx + c$, where m and c are constants, connects two variables x and y. In such a case we say that y is partly constant and partly varies as x.

Example 7

A chicken dealer buys chicks at sh. 90 each from a market. It costs him sh. 520 to transport chicks.

(a) Find the total amount he spends if he buys 15 chicks.

(b) If he spends a total of sh. c to buy and transport n chicks, write an equation connecting c and n.

Solution

(a) Total amount spent comprises the transport charges and the cost of 15 chicks.

$$\begin{aligned} \therefore \text{ Total cost} &= 520 + 90 \times 15 \\ &= 520 + 1\ 350 \\ &= \text{sh. } 1\ 870 \end{aligned}$$

(b) Total cost is sh. c
Cost of 90 chicks is sh. 90n
Transport charges is sh. 520
$\therefore$ C $= 520 + 90n$

Note that the total cost is the sum of a fixed amount (constant) and a variable amount which depends on the number of chicks bought.

Example 8

A variable y is partly constant and partly varies as x. If $x = 2$ when $y = 7$ and $x = 4$ when $y = 11$, find the equation connecting y and x.

Solution

The required equation can be written in the form;
$y = kx + c$ where k and c are constants.
Substituting $x = 2$, $y = 7$ and $x = 4$, $y = 11$ in the equation gives;
$7 = 2k + c$ ....(1)
$11 = 4k + c$ ....(2)
Subtracting equation (1) from equation (2);
$4 = 2k$
$\therefore k = 2$
Substituting $k = 2$ in equation (1);
$c = 7 - 4$
$c = 3$
$\therefore$ The required equation is $y = 2x + 3$.

Example 9

The cost (c) of hiring a school bus consists of two parts, one of which is fixed and the other varies as the distance (D) covered by the bus. If sh. 4 500 is charged for hiring the bus for a distance of 100 km and sh. 4 000 for a distance of 60 km, find an equation connecting c and d.

Solution

The equation is $c = a + bd$ where a and b are constants.
Subsituting the values $c = 4\,500$, $d = 100$ and $c = 4\,000$, $d = 60$
$4\,500 = a + 100b$ ....(1)
$4\,000 = a + 60b$ .... (2)
Subtracting equation (2) from equation (1);
$500 = 40b$
$b = 12.5$

Substituting this in equation (2);

$a = 4\,000 - 60 \times 12.5$

$= 4\,000 - 750$

$= 3\,250$

∴ The equation is $c = 3\,250 + 12.5d$

Note:

In an equation such as:

(i) $y = ax + bx^2$ where a and b are constants, y varies partly as x and partly as the square of x.

(ii) $y = ax + \frac{b}{z}$ where a and b are constants, y varies partly as x and partly as the inverse of z.

Exercise 9.4

1. A quantity y is partly constant and partly varies as x. If $x = 4$ when $y = o$ and $x = 12$ when $y = 40$, find the equation connecting y and x.
2. P is partly constant and partly varies inversely as Q. If $Q = 9$ when $P = 3$ and $Q = 18$ when $P = 9$, find P when $Q = 12$.
3. The speed v ms^{-1} of a moving particle is partly constant and partly varies as time t seconds. It is given that $v = 12$ MS^{-1} when $t = 0$ and $V = 60$ ms^{-1} when $t = 8$ seconds. Find:
   (a) an equation connecting V and t.
   (b) the speed of the particle when the time is 6 seconds.
4. The cost c of renting a car includes a fixed charge of sh. 2 000 and a variable charge of sh. 50 per km. Find the cost of renting a car and driving it for 750 km. Write an equation connecting c and s, where s is the number of kilometres travelled.
5. The resistance to the motion of a bicycle is partly a constant and partly varies as the square of the speed. The resistance is 265 N when the speed is 20 km/h and 365 N when the speed is 30 km/h. Find the resistance when the speed is 35 km/h.
6. A quantity y is partly constant and partly varies inversely as x^2. If $y = 7$ when $x = 10$ and $y = 5\frac{1}{2}$ when $x = 20$, write an equation connecting y and x. Hence, find y when $x = 18$.
7. A quantity y is partly constant and partly varies inversely as x^3. If $y = 12.3$ when $x = 10$ and $y = 12\frac{3}{80}$ when $x = 20$, write an equation connecting y and x. Hence find y when $x = 3$.
8. y varies partly as x and partly as the square of x. When $x = 2$, $y = 14$ and

when $x = 5$, $y = 65$. Express y in terms of x. Hence find y when $x = -2$.

9. y varies partly as the inverse of x and partly as x. When $x = 1$, $y = 1$ and when $x = 4$, $y = -11$. Determine the equation connecting y and x. Find the value of y when $x = 2$.

Joint Variation

The formula for simple interest I is $I = \frac{PRT}{100}$. The interest I varies directly as each of the quantities P, R and T. The constant of proportionality is $\frac{1}{100}$. A change in any one of the quantities P, R or T causes a proportionate change in I (assuming that the other two remain constant). This is an example of a joint variation. Other examples of joint variations are:

(i) the volume (V) of a cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder. In this case, the volume (V) of a cylinder varies jointly as the height h and the square of the radius r of the base.

(ii) the force F of attraction between two bodies of masses (M and m) which are a given distance d apart is given by the formula $F = \frac{kMm}{d^2}$.

In this case, F varies jointly as M and m and inversely as the square of d.

Example 10

P varies jointly as Q and the square of R. $P = 18$ when $Q = 9$ and $R = 15$. Find R when $P = 32$ and $Q = 81$.

Solution

$P \propto QR^2$

So, $P = kQR^2$, where k is a constant.

Substituting $P = 18$, $Q = 9$ and $R = 15$;

$18 = k \times 9 \times 15^2$

$\therefore\ k = \frac{18}{9 \times 225}$

$= \frac{2}{225}$

The equation is, therefore, $P = \frac{2}{225} QR^2$

When $P = 32$ and $Q = 81$;

$32 = \frac{2}{225} \times 81R^2$

$R^2 = \frac{32 \times 225}{2 \times 81}$

$= \frac{400}{9}$

$$R = \sqrt{\frac{400}{9}}$$

$$= \frac{20}{3}$$

$$= 6\tfrac{2}{3}$$

Example 11

A varies directly as B and inversely as the square root of C. Find the percentage change in A when B is decreased by 10% and C increased by 21%.

Solution

$$A = k\frac{B}{\sqrt{C}} \quad \text{............(1)}$$

A change in B and C causes a change in A.

$$A_1 = k\frac{B_1}{\sqrt{C_1}} \quad \text{................(2)}$$

$$B_1 = \frac{90}{100}B$$

$$= 0.9B$$

$$C_1 = \frac{121}{100}C$$

$$= 1.21C$$

Substituting B_1 and C_1 in equation (2);

$$A_1 = k\frac{0.9\,B}{\sqrt{1.21\,C}}$$

$$= \frac{0.9}{1.1}\left(k\frac{B}{\sqrt{C}}\right)$$

$$= \frac{9}{11}A$$

$$\text{Percentage change in A} = \frac{A_1 - A}{A} \times 100\%$$

$$= \frac{\frac{9}{11}A - A}{A} \times 100\%$$

$$= -18\tfrac{2}{11}\%$$

$\therefore$ A decreases by $18\tfrac{2}{11}\%$

Exercise 9.5

1. Write a formula connecting the variables in each of the following:
   (a) A varies jointly as b and h.
   (b) A varies jointly as h and the cube of r.
   (c) y varies directly as x and inversely as z.
   (d) w varies directly as b and as the square of d and inversely as *l*
   (e) H varies directly as t and the square of E and inversely as R.
   (f) p varies directly as the square root of t and inversely as the square root of v.
2. A quantity y varies directly as quantity x and inversely as the cube of a quantity z. If x = 12 when z = 2 and y = $\frac{1}{24}$, find the formula connecting x, y and z.
3. The mass of a certain wire varies jointly as its length and the square of its diameter. If 750 m of wire of diameter 2.5 mm has a mass of 45 kg, what will be the mass of 1.5 km of the wire of diameter 4.5 mm?
4. It is given that $P \propto QR^2$. When Q = 9, R = 6 and P = 13.5. Find P when Q = 12 and R = 15. Find R when P = 43 and Q = 10.5. If R is multiplied by 3 and Q is divided by 3, by what factor is P multiplied?
5. Given that $m \propto r^2 h$. Find the percentage change in m if r is decreased by 15% and h is increased by 12%.
6. Given that $P \propto QR$ and that Q = 12, R = 27 when P = 18, calculate:
   (a) the value of P when Q = 9 and R = 30.
   (b) the value of R when P = 60 and Q = 45.
   (c) the percentage by which P is changed when Q is decreased by 12% and R increased by 12%.
7. If $y \propto x$ and $x \propto t^2$, determine how y varies with t.
8. Given that T varies directly as the square of x and inversely as y and that y varies directly as the product of T and x, determine how T varies with x.
9. A quantity y varies directly as x and inversely as the square of z. If y = 4 when x = 2 and x = 5, find the equation connecting y, x and z.
10. The volume V cm^3 of a fixed mass of gas varies directly as it absolute temperature T kelvin and pressure (P) mmHg. If 1g of the gas has a volume of 1 cm^3 at a temperature of 92^{0}C and pressure of 13.6 mmHg, find its volume at 72^{0}C and 100 mmHg.

Chapter Ten

SEQUENCES AND SERIES

10.1: Sequences

Study the following patterns of numbers:

(i) $2, 4, 6, 8 \ldots$

(ii) $1, 4, 9, 16 \ldots$

(iii) $1, {}^{-}2, 4, {}^{-}8 \ldots$

In each case, state the next two numbers. How did you obtain your answers?

Each of the missing numbers is obtained using some rule. If numbers are arranged in a definite order (according to a specific rule), they form a sequence. All the three patterns above are examples of sequences. Each of the numbers in the sequence is called a term. In sequence (i) above, the first term is 2, the third term is 6 and the fifth term is 10. State the first, third and fifth terms in sequences (ii) and (iii).

Example 1

For each of the following sequences, find the next three terms:

(a) $1, 3, 5, 7 \ldots$

(b) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$

(c) $1, 8, 27 \ldots$

(d) ${}^{-}3, 6, {}^{-}12 \ldots$

Solution

(a) To obtain the next term, add 2 to the preceding term. The next three terms are 9, 11 and 13.

(b) To obtain the next term, the preceding term is multiplied by $\frac{1}{3}$. The next three terms are $\frac{1}{81}$, $\frac{1}{243}$ and $\frac{1}{729}$.

(c) To obtain the n^{th} term, n is cubed. The next three terms are 64, 125 and 216.

(d) The next term is obtained by multiplying the preceding term by –2. The next three terms are 24, –48 and 96.

Example 2

For the n^{th} term of a sequence is given by $2n + 3$, find the first, fifth and twelfth terms.

Solution

For the first term, n = 1. Substituting, $2 \times 1 + 3 = 5$

In the fifth term, n = 5. Substituting, $2 \times 5 + 3 = 13$

In the twelfth term n = 12. Substituting, $2 \times 12 + 3 = 27$

Given a sequence, it is possible to deduce a general rule (n^{th} term) for the sequence.

Example 3

Deduce the general rule in the sequence given below; 2, 5, 8, 11 ...

Solution

Position	1	2	3	4
Term	2	5	8	11

Notice that;

$2 = 1 \times 3 - 1$

$5 = 2 \times 3 - 1$

$8 = 3 \times 3 - 1$

$11 = 4 \times 3 - 1$

Hence, the n^{th} term is; $n \times 3 - 1 = 3n - 1$

Exercise 10.1

1. For each of the following sequences, find the terms indicated:
   (a) 1, 2, 3, 4 ... 6th and 20th terms.
   (b) 2, 5, 10, 17 ... 6th term.
   (c) 3, 7, 11, 15 ... 7th and 10th terms.
   (d) 0, 3, 8, 15 ... 5th and 8th terms.
   (e) 21, 15, 9, 3 ... 5th and 9th terms.
   (f) $-15, -12, -9, -6$ ... 7th and 11th terms.
   (g) $\frac{1}{2}, 1, 2$ ... 4th and 5th terms.
   (h) 0, 2, 6, 12 ... 7th and 12th terms.
   (i) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$ ... 6th and 10th terms.
   (j) $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ ... 8th and 10th terms.
   (k) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}$ ... 7th and 9th terms.
   (l) $\frac{1}{5}, 1, \frac{9}{5}$ ... 5th and 8th terms.
2. Write down the first three terms of the sequence whose n^{th} term is given below:

(a) $2n$ (b) $2n + 1$ (c) n^2

(d) $\frac{n^2 + 1}{2}$ (e) $2n^2 - 2$ (f) $2^n + 1$

3. Give an expression for the nth term:
   - (a) 3, 6, 9, 12 ...
   - (b) 1, 3, 5, 7, 9...
   - (c) 6, 11, 16 ...
   - (d) 0, 3, 8, 15 ...

10.2: Arithmetic and Geometric Sequences

Arithmetic Sequences

In the sequence 1, 4, 7, 10, the next term is obtained by adding 3 to the preceding one. 3 is called the common difference (d) of the sequence. It is the difference between any two consecutive terms in the sequence.

Any sequence of numbers which have a common difference is called an Arithmetic sequence. The general form of an arithmetic sequence is a, a + d, a + 2d, etc, a + 3d, where a is the first term and d is the common difference.

Consider the sequence given by $3 + 2n$. The first term is when $n = 1$ and is $3 + 2 \times 1 = 5$. The second term is when $n = 2$ and is $3 + 2 \times 2 = 7$, and so on. The sequence is 5, 7, 9, 11 .... The nth term is given by the first term 5 added to the product of the common difference (2) and $(n - 1)$.

In general, an arithmetic sequence with the first term (a) and common difference (d) will have the sequence below.

1st	a
2nd	$a + d$
3rd	$a + 2d$
4th	$a + 3d$
nth	$a + (n - 1) d$

The nth term of an arithmetic sequence is given by $a + (n - 1)d$.

Example 4

Given the arithmetic sequence 4, 11, 18..., find:

(a) the common difference.

(b) the 6th term.

Solution

(a) The common difference $d = 11 - 4$

$= 7$

(b) The nth term $= a + (n - 1)d$

$\therefore$ 6th term $= 4 + (6 - 1)\,7$
$= 39$

Example 5

The 20th term of an arithmetic sequence is 60 and 16th term is 20. Find the first term and the common difference.

Solution

$a + (20 - 1)d = 60$
$\therefore a + 19d = 60 \ldots\ldots\ldots (1)$
$a + (16 - 1)d = 20$
$\therefore a + 15d = 20 \ldots\ldots\ldots\ldots (2)$
(1) – (2) gives;
$4d = 40$
$d = 10$
But $a + 15d = 20$
$\therefore a + 15 \times 10 = 20$
$a + 150 = 20$
$a = -130$
Hence, the first term is –130 and the common difference is 10.

Example 6

Find the number of terms in the sequence –3, 0, 3 …54

Solution

The nth term is $a + (n - 1)d$.
$a = -3, d = 3$
nth term $= 54$
$\therefore -3 + (n - 1)3 = 54$
$3(n - 1) = 57$
$n - 1 = 19$
$n = 20$

Exercise 10.2

1. State the 5th term of the sequence 5, 9, 13 ….
2. State the 6th and 10th terms of the sequence –7, –4, –1, 2 ….
3. In the sequence 3, –1, –5, –9 …, state:
   (a) the common difference.
   (b) the 5th term.

4. In an arithmetic sequence, the 15th term is 52, and the common difference is 3. Find:
   (a) the first term.
   (b) the 20th term.
5. The second term of an arithmetic sequence is 7 and the ninth term 28. Find the common difference and the first term of the sequence.
6. The 10th term of an arithmetic sequence is 42 more than the 7th term. If the first term is −14, find the second term of the sequence.
7. Find the number of terms in the sequence 3, 7, 11 … 83.
8. The product of the third and second term of an arithmetic sequence is zero. Find the first term if the common difference is 10.
9. In January 1990, a man's salary was K£ 2 520 p.a. If his annual increment is K£ 108, find what his salary was in January 1996.
10. The product of the third and fourth terms of an arithmetic sequence is 3 000. Find the first term if the common difference is 10.
11. If the difference between the 7th and the 9th terms of an arithmetic sequence is 25, find the common difference of the sequence.
12. The first term of an arithmetic sequence in 4 and the last term is 64. If the common difference is 5, find the number of terms.
13. The average of the second and the third terms of an arithmetic sequence is 4. If the first term is −2, find the 6th term.
14. A businesswoman wants to raise her capital to sh. 20 000. She opens an interest-free account with initial deposit of sh. 1 500 in the first month. She decides to deposit sh. 500 per month. After how many years will she be able to raise the capital?

Geometric Sequence

Consider the sequence 1, 2, 4, 8, 16 …. It can be seen that in this sequence, each term is obtained by multiplying the preceding one by a constant number 2. In other words;

$$\frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$$

The constant number 2 is called the **common ratio**. A sequence with a common ratio is known as **geometric sequence**.

Which of the following are geometric sequences? For the geometric sequences, what is the common ratio?

(i) 1, 3, 9, 27, 81 …

(ii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

(iii) 9, 13, 17, 21, 25 …

(iv) 8, 6, 4, 2 ...

(v) –4, 8, –16, 32 ...

Consider a general case with the first term a and the common ratio r.
The sequence is a, ar, ar^2, ar^3 ....
The first term is a.
The second term is ar = ar^{2-1}.
The third term is $ar^2 = ar^{3-1}$, etc.

Generally, the n^{th} term of a geometric sequence with a as the first term and r as the common ratio is ar^{n-1}. The sequence is a, ar, ar^2 ....

Example 7

Given the geometric sequence 4, 12, 36 ..., find the 4th, 5th and the nth terms.

Solution

The first term, a = 4
The common ratio, r = 3
Therefore, the 4th term = $4 \times 3^{4-1}$
$= 4 \times 3^3$
$= 108$
The 5th term $= 4 \times 3^{5-1}$
$= 4 \times 3^4$
$= 324$
Then nth term $= 4 \times 3^{n-1}$

Example 8

The 4th term of a geometric sequence is 16. If the first term is 2, find:

(a) the common ratio.
(b) the seventh term.

Solution

(a) The first term, a = 2
The 4th term is $2 \times r^{4-1} = 16$
Thus, $2r^3 = 16$
$r^3 = 8$
∴ The common ratio is 2.

(b) The 7th term = $ar^6 = 2 \times 2^6$
$= 128$

Example 9

The fourth term of a geometric sequence is 8 and the 6th term 32. Find the two possible common ratios.

Solution

The 4th term is $ar^3 = 8$

The 6th term is $ar^5 = 32$

$$\therefore \frac{ar^5}{ar^3} = \frac{32}{8}$$

$$r^2 = 4$$

$$\therefore r = \pm 2$$

The possible common ratios are r = 2 and r – 2. Which are the two possible sequences?

Exercise 10.3

1. Write down the common ratio in each of the following:
   (a) $4, 20, 100 \ldots$ (b) $3, -6, 12 \ldots$ (c) $4, \frac{1}{2}, \frac{1}{16} \ldots$
   (d) $-2, \frac{1}{2}, -\frac{1}{8} \ldots$ (e) $c, ca, ca^2 \ldots$ (f) $b, -b, b, -b \ldots$
   (g) $m, -mn, mn^2 \ldots$
2. For each of the following geometric sequences, find the terms indicated:
   (a) $2, 4, 8 \ldots$ 6th and 8th terms.
   (b) $32, -16, 8 \ldots$ 8th and nth terms.
   (c) $1, \frac{1}{2}, \frac{1}{4} \ldots$ 7th and nth terms.
   (d) $1, -1, 1 \ldots$ 30th and 79th terms.
   (e) $27, 9, 3 \ldots$ 8th and nth terms.
   (f) $b, \frac{b}{2}, \frac{b}{4} \ldots$ 9th term.
3. The ratio of the 10th to the 8th term of a geometric sequence is 9. Find the two possible common ratios.
4. The third term of a geometric sequence is 1. If the sixth of the same sequence is $\frac{1}{27}$, write down the first four terms of the sequence.
5. The ratio of the fourth to the first term of a geometric sequence is $\frac{1}{8}$. If the first term exceeds the second by 5, find the first and the 8th terms of the sequence.
6. The first term of a geometric sequence is $x + 1$. If the third term of the same sequence is $(x + 1)(x^2 - 2x + 1)$, show that the second term is $x^2 - 1$.
7. The ratio of the 6th to the 2nd term of a geometric sequence is 256. If the 3rd term of this sequence is 32, determine its first and fifth terms.
8. If $(a + 1), (a + 1)x^2, (a + 1)x^4 \ldots$ is a geometric sequence, write down expressions for the nth and 9th terms.
9. The first, third and fifth terms of a geometric sequence form arithmetic

sequence. If the first term of the sequence is 3, find the 10th term of the geometric sequence.

10. The current value of a vehicle is sh. 128 000. If this vehicle has been depreciating by $\frac{1}{5}$th of its value every year, find its value four years ago.

10.3: Series

The sum of the terms of a sequence is called a **series**. If the terms of the sequence 1, 4, 7, 10 ... are written with the addition sign, we obtain the series;
$1 + 4 + 7 + 10 + \ldots$

Arithmetic Series/Arithmetic Progression, A.P.

This series is obtained by adding an arithmetic sequence.
Consider the series $2 + 5 + 8$
Let S^n stand for the sum of the first n terms of the series.
Thus, $S_1 = 2$
$S_2 = 2 + 5 = 7$
$S_3 = 2 + 5 + 8 = 15$
$S_6 = 2 + 5 + 8 + 11 + 14 + 17 = 57$
We can write S_6 in the following way;

$$S_6 = 2 + 5 + 8 + 11 + 14 + 17 \ldots\ldots\ldots\ldots (1)$$

Or $$S_6 = 17 + 14 + 11 + 8 + 5 + 2 \ldots\ldots\ldots\ldots (2)$$

Adding (1) and (2);

$$2S_6 = 19 + 19 + 19 + 19 + 19 + 19$$
$$= 19 \times 6$$
$$\therefore S_6 = \frac{19 \times 6}{2}$$
$$= 57$$

Use the same to find the sum of the first 15 terms of the series.
In general, for an, A.P.;

$$S_n = a + (a + d) + (a + 2d) + \ldots [a + (n - 1) d] \ldots\ldots\ldots (1)$$

Also $S_n = [a + (n - 1)d] + \ldots (a + 2d) + (a + d) + a \ldots\ldots\ldots (2)$

Adding (1) and (2);

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] \ldots + [2a + (n - 1) d]$$
$$= n [2a + (n - 1)d]$$
$$\therefore \quad S_n = \frac{n}{2} [2a + (n - 1)d]$$

If the first term (a) and the last term l are given, then $S_n = \frac{n}{2} (a + l)$

Example 10

In the arithmetic series 1 + 4 + 7 + 10 + …, find the sum of the first

(a) 10 terms.

(b) 100 terms.

Solution

(a) $a = 1, d = 3$ and $n = 10$

$\therefore S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{10} = \frac{10}{2}(2 \times 1 + (10 - 1)3]$

$= 5[2 + 27]$

$= 5 \times 29$

$= 145$

(b) $a = 1, d = 3$ and $n = 100$

$\therefore S_{100} = \frac{100}{2}[2 \times 1 + (100 - 1)3]$

$= 50[2 + 99 \times 3]$

$= 50[2 + 297]$

$= 50 \times 299$

$= 14\,950$

Example 11

The sum of the first eight terms of an arithmetic Progression (A.P.) is 220. If the third term is 17, find the sum of the first six terms.

Solution

$S_8 = \frac{8}{2}[2a + (8 - 1)d]$

$= 4(2a + 7d)$

So, $8a + 28d = 220$ ……… (1)

The third term is $a + (3 - 1)d = a + 2d = 17$ …….. (2)

Solving (1) and (2) simultaneously;

$8a + 28d = 220$ (i)

$8a + 16d = 136$ 8x ……….. (2)

$\therefore 12d = 84$

$d = 7.$

Substituting $d = 7$ in equation (2) gives $a = 3$.

Therefore, $S_6 = \frac{6}{2}[2 \times 3 + (6 - 1)7]$

$= 3(6 \times 35)$

$= 3 \times 41$

$= 123$

Exercise 10.4

1. Find the sum of the number of terms in the following A.P. s:
   (a) 2 + 4 + 6 … the first 8 terms
   (b) 5 + 8 + 11 …the first 6 terms

(c) $11 + 8 + 5 \ldots$ the first 5 terms
(d) $3 + 10 + 17 \ldots$ the first 10 terms
(e) $-9 - 12 - 15 \ldots$ the first 14 terms
(f) $-9 - 6 - 3 \ldots$ the first 7 terms
(g) $10 + 13 + 16 \ldots$ the first 20 terms
(h) $30 + 35 + 40 \ldots$ the first 10 terms
(i) $20 + 29 + 38 \ldots$ the first 51 terms
(j) $1 + \frac{3}{2} + 2 + \frac{5}{2} \ldots$ the first 11 terms

2. Find the sum of the A.P.s. below:
   (a) $2 + 4 + 6 \ldots + 42$
   (b) $3 + 6 + 9 \ldots + 27$
   (c) $4 + 11 + 18 \ldots + 39$
3. The sum of the first 10 terms of an arithmetic series is 400. If the sum of the first 6 terms of the same series is 120, find the 15th term.
4. The sum of the first 20 terms of an arithmetic series is $7\frac{1}{2}$. If the third term of the series is 2, find the sum of the first 13 terms.
5. Find the sum of the first 11 terms of an arithmetic series, if the first term is 3 and the common difference is 6.
6. The sum of the first 6 terms of an arithmetic series is 51. If the first term is -4, find the common difference.
7. An arithmetic series has a common difference as $-\frac{1}{4}$ and first term as 3. Find the number of terms of the series which would give a sum of zero.
8. Two arithmetic series are such that their common differences are 9 and 3 respectively. If their first terms are 2 and 5 respectively, find the number of terms of each series that would give a common sum.
9. The sum of the first four terms of an arithmetic series is 46. If the sum of the first 10 terms is 25, find the first term and the common difference.
10. A man buys premium bonds every year. In the first year, he buys sh. 2 000 worth of bonds. If every year he increases his annual investment in bonds by sh. 600, how long will he take for his total investment in bonds to be sh. 37 000.

Geometric Series

The series obtained by adding the terms of a geometric sequence is called **geometric series** or **geometric progression** (G.P.)

Consider the series $2 + 6 + 18 + 54 + \ldots$

$S_1 = 2$
$S_2 = 2 + 6 = 8$
$S_4 = 2 + 6 + 8 + 54 = 80$

$S_6 = 2 + 6 + 8 + 54 + 162 + 486 = 728$

We can write S_6 in the following way;

$S_6 = 2 + 6 + 8 + 54 + 162 + 486$ .... (1)

Multiplying (i) by common ratio 3;

$3S_6 = 6 + 18 + 24 + 162 + 486 + 1458$ .... (2)

Subtracting (1) from (2);

$3S_6 - S_6 = 1\,458 - 2$

$\therefore S_6 = \frac{1\,456}{2}$

$= 728$

Use the same method to find the sum of the first 50 terms of the G.P.

In general, for a G.P.;

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ .... (1)

Multiplying (1) by the common ratio (r)

$rSn = ar + ar^2 + ar^3 + \ldots + ar^n$ .... (2)

Subtracting (1) from (2);

$rS_n - S_n = ar^n - a$

$S_n(r - 1) = a(r^n - 1)$

$S_n = \frac{a(r^n - 1)}{r - 1}$ .... (3)

Alternatively;

$S_n - rS_n = a - ar^n$

$S_n(1 - r) = (1 - r^n)$

$S_n = \frac{a(1 - r^n)}{1 - r}$ .... (4)

The formulae (3) and (4) above represent the sum of the first n terms of a geometric series with first term a and common ratio r. Formula (3) is convenient where $|r| > 1$ while (4) in preferable when $|r| < 1$

Example 12

Find the sum of the first 9 terms of the G.P. $8 + 24 + 72 + \ldots$

Solution

$a = 8,\ r = \frac{24}{8} = 3$

$S_n = \frac{a(r^n - 1)}{r - 1}\ |r| > 1$

$= \frac{8(3^9 - 1)}{3 - 1}$

$= \frac{8(19\,683 - 1)}{2}$

$= 4 \times 19\,682$

$= 78\,728$

Example 13

In the G.P. 8 – 4 + 2 – + …, find the sum of the first 10 terms.

Solution

$A = 8, r = -\frac{4}{8} = -\frac{1}{2}$

$S_n = \frac{a(1 - r^n)}{1 - r}$ $|r| < 1$

$$\therefore\ S_{10} = \frac{8(1 - (-\frac{1}{2})^{10})}{1 - \frac{-1}{2}}$$

$$= \frac{8(1 - \frac{1}{1024})}{\frac{3}{2}}$$

$$= \frac{16}{3}\left(\frac{1\,023}{1\,024}\right)$$

$$= \frac{341}{64}$$

$$= 5\frac{21}{64}$$

Example 14

The sum of the first three terms of a geometric series is 26. If the common ratio is 3, find the sum of the first six terms.

Solution

$S_3 = 26,\ r = 3\ \ n = 3$

$S_n = \frac{a(r^n - 1)}{r - 1}$

Substituting;

$$26 = a\frac{(3^3 - 1)}{3 - 1}$$

$$26 = a\frac{(27 - 1)}{2}$$

$$a = \frac{26 \times 2}{26}$$

$$= 2$$

$$S_6 = \frac{2(3^6 - 1)}{26}$$

$$= \frac{2 \times 728}{2}$$

$$= 728$$

Exercise 10.5

1. Find the sum of the number of terms indicated in each of the following G.P.s.:
   (a) $2 + 4 + 8 + 16 \ldots$ (8 terms)
   (b) $8 + 24 + 72 \ldots$ (8 terms)
   (c) $4 - 12 + 36 \ldots$ (8 terms)
   (d) $27 + 9 + 3 \ldots$ (7 terms)
   (e) $\frac{1}{62} + \frac{1}{31} + \frac{2}{31} \ldots$ (5 terms)
   (f) $9 - 6 + 4 \ldots$ (5 terms)
   (g) $-4 - 2 - 1 \ldots$ (8 terms)
   (h) $10 + 20 + 40 + 80 \ldots$ (14 terms)
   (i) $-3 + 3 - 3 + 3 \ldots$ (25 terms)
   (j) $18 - 6 + 2 \ldots$ (10 tems)
2. The sum of the first 3 sums of a geometric series is 31. If its common ratio is $\frac{1}{5}$, calculate the sum of the first 8 term of this series.
3. The first term of a geometric series is 4. What is the sum of the first 6 terms of the series if its common ratio is 3?
4. The sum of the first three terms of a geometric sequence is 14. If the common ratio is -3, find the sum of the first five terms.
5. In a geometric series, the first term is 2 and the common ratio is 3. Find the number of terms that will give a sum of 242.
6. The fourth term of a geometric sequence is 192. If the first term of the sequence is 3, find:
   (a) the common ratio.
   (b) the number of terms that will give a sum of 255.
7. A geometric series is such that its first term is 2. Find the two possible common ratios if the sum of its first three terms is 26.
8. The sum of the first three terms of a geometric series is 35. If the common ratio is 2, what is the first term?
9. The first term of a G.P. is 6 and the sum of the first three terms is 42. Find the value of the common ratio.
10. In a G.P. the sum of the 2nd and 3rd is 4 and the sum of the 3rd and sum of 3rd and 4th terms is 2. Find the first term and the common ratio.
11. The first term of a G.P. is 4. If the common ratio is 2, find the greatest number of terms that will give a sum less than 40.

12. What is the least number of terms of the G.P. 2 + 4 + 8... that will give a sum greater than 1 500 000.
13. A ball allowed to drop from a height of 3 m on to a horizontal ground rebounds to $\frac{3}{4}$ of its previous height. Find, to the nearest metre, the total distance the ball will have travelled when it hits the ground for the 8th time.
14. Mutiso's salary is K£ 12 000 p.a. His salary increases by 10% annually. Find the total amount he will have earned in six years.
15. The 2nd, 4th and 7th terms of an A.P. are the first 3 consecutive terms of a G.P. Find:
    (a) the common ratio.
    (b) The sum of the first eight terms of the G.P. if the common difference of the A.P. is 2.

Chapter 11

VECTORS (II)

11.1: Co-ordinates in Two and Three Dimensions

In chapter 19 of Book one, we described points in a plane (two dimensions) using the rectangular cartesian system (x, y). In this chapter, we shall describe points in three dimensions. Figure 11.1 is a wire model of a cuboid with OABC as its base.

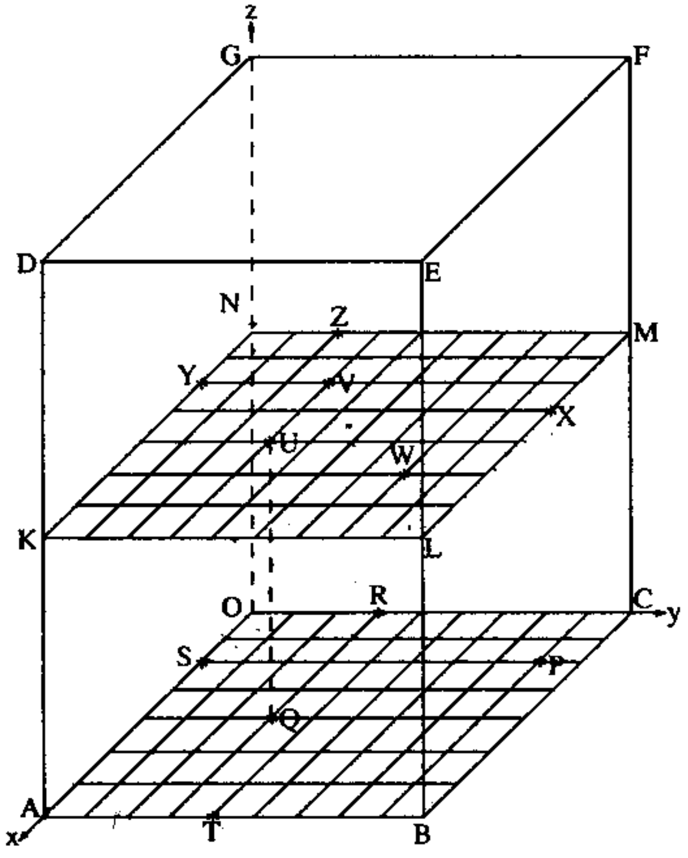


Fig. 11.1

If we take O as the origin, OA as the x-axis and OC as the y-axis, the co-ordinates of any point in the plane OABC can be determined in two dimensions. For example, the co-ordinates of Q are (4, 3). What are the co-ordinates of P, R and S?

Suppose we now wish to describe a point that is not on the plane OABC. We need a third axis which is perpendicular to both x and y-axes. This third axis is conventionally denoted by the letter z.

Thus, point U which is 7 units above Q has the co-ordinates (4, 3, 7) in three dimensional co-ordinate system. Similarly, the co-ordinates of Q are (4, 3, 0) in three dimensional co-ordinate system. Given that the plane KLMN is 7 units above the base and the height of the model is 14 units, then the co-ordinates of:

(i) V are (2, 3, 7)
(ii) P are (2, 8, 0)
(iii) D are (7, 0, 14)

What are the co-ordinates of S, W, F, Y and R?

Exercise 11.1

1. Using figure 11.1, what are the co-ordinates of:
   (a) T, X and Z?
   (b) the vertices of the rectangles OABC, KLMN and DEFG?
   (c) the centre of rectangle DEFG?
   (d) the intersection of BF and EC?
2. In figure 11.2, OC is the x-axis , OB the y-axis and LG the z-axis.

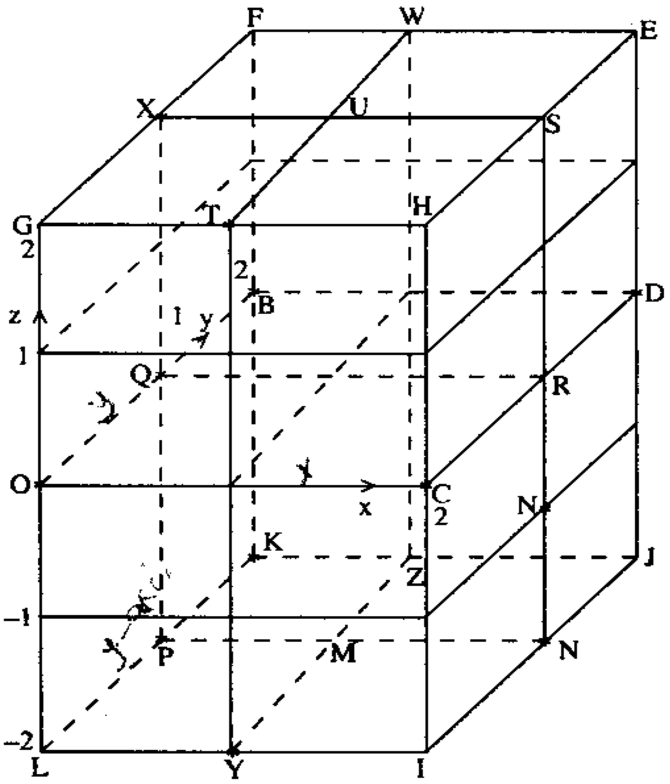


Fig. 11.2

State the points with the following co-ordinates:

(a) $(0, 0, -2)$ (b) $(1, 1, -2)$ (c) $(2, 2, 2)$
(d) $(1, 0, 2,)$ (e) $(2, 2, 0)$ (f) $(2, 1, -1)$

3. State the axis on which each of the following points lies:

(a) P(6, 0, 0) (b) Q (0, −3, 0) (c) R(0, 0, 12)

4. ABCD is the floor of a rectangular room. AB is 8 m, AD 12 m and the room 6 m high. By taking AB as the x-axis, AD as the y-axis and one metre to represent one unit, give the co-ordinates of:
   (a) a fly which is 4 metres above the floor, 2 metres from the vertical wall through AB and 1 metre from the vertical wall through AD.
   (b) a mosquito resting on the ceiling of the room at a point vertically above the centre of the floor.
   (c) the corner of the room farthest from A.
5. Two roads running due north and east meet at R. An aircraft is located $1\frac{3}{4}$ km above point A on the ground. The point A is 8 kilometres from R on a bearing of 045^0. Taking eastwards as the positive x direction and northwards as the positive y direction, give the co-ordinates of the position of the aircraft.

Use figure 11.1 to answer questions 6-8.

6. The co-ordinates of the image of U (4, 3, 7) after a reflection in the x-y plane are (4, 3, −7). State the co-ordinates of the images of the following points after reflection in the x-y plane: V, X, M, R, E and G.
7. Reflect all points given in questions 6 in the y-z plane and write down the co-ordinates of their images.
8. Write down the co-ordinates of images of the following points after reflection in the x-z plane; W, E, Q, Y ...and M.
9. If Q (a, b, c) is reflected in the x-y plane, the co-ordinates of the image are (a, b, -c). Write down the co-ordinates of Q (a, b, c) after reflection in the
   (a) y-z, and,
   (b) x-z plane.

11.2: Column and Position Vectors in Three Dimensions

In Book two, we learnt that a displacement in a plane is represented by a column vector of the form $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$. For example, the displacement from (2, 1) to (4, 7) is represented by the column vector; $\begin{pmatrix} 4-2 \\ 7-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

Similarly, in three-dimensions, a displacement is represented by a column vector of the form $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$, where p, q and r are the changes in x, y, z directions respectively.

For example, the displacement from A (3, 1, 4) to B (7, 2, 6) is represented by the column vector; $\begin{pmatrix} 7-3 \\ 2-1 \\ 6-4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

In the example above, the position vector of A written as **OA** is $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, where O is the origin. What is the position vector of B?

Addition of vectors in three dimensions is done in the same way as that in two dimensions. Scalar multiplication is also similar.

For example, if $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$

then (i) $3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$ (ii) $4\mathbf{a} - \frac{1}{2}\mathbf{b} = 4\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ -6 \\ 15 \end{pmatrix} + \begin{pmatrix} -4 \\ 16 \\ 20 \end{pmatrix}$ $= \begin{pmatrix} 12 \\ -8 \\ 20 \end{pmatrix} + \begin{pmatrix} 1 \\ -4 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 10 \\ 35 \end{pmatrix}$ $= \begin{pmatrix} 13 \\ -12 \\ 15 \end{pmatrix}$

Exercise 11.2

1. If $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$, find:
   (a) $\mathbf{a} + \mathbf{b} - 2\mathbf{c}$ (b) $3\mathbf{a} + \frac{7}{8}\mathbf{b} + \frac{5}{6}\mathbf{c}$
   (c) $3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ (d) $5\mathbf{a} - 0.7\mathbf{b} + 0.3\mathbf{c}$
2. Find the column vectors **AB** given the points:
   (a) A (5, 1, 1) and B (7, 4, 2)
   (b) A (–1, 0, 2) and B (1, –4, 7)
   (c) A (–3, –1, –2) and B (3, 1, 2)
   (d) A (0, 2, 3) and B (–4, 1, –2)
   (e) A (–2, 1, 5) and B (–4, –1, –2)
3. A bird flies from a point P to a point Q through the following displacement

vectors; $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 9 \\ 1 \\ 10 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 8 \\ 10 \end{pmatrix}$. Taking P as the origin, determine the co-ordinates of Q.

Use figure 11.1 to answer question 4 and 5.

4. What are the position vectors of W, Q, P, X, G and D?
5. Determine the position vectors of H, given that:

   (a) $\mathbf{WH} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ (b) $\mathbf{HW} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$ (c) $\mathbf{AH} = \begin{pmatrix} -3 \\ 3 \\ 7 \end{pmatrix}$

   (d) $\mathbf{PH} = \begin{pmatrix} 0 \\ 0 \\ 14 \end{pmatrix}$ (e) $\mathbf{TH} = \begin{pmatrix} -7 \\ 5 \\ 14 \end{pmatrix}$

6. Figure 11.3 shows a rectangular box in which ABCD is the base, AB = 50 cm, AD = 60 cm and AH = 72 cm. O is the centre of the base. P is the midpoint of BC, Q is such that CQ : QF = 1 : 2, R is 10 cm below F, S is 10 cm from F, T is 10 cm from EF and EH and on the plane EFGH. A cockroach moves from point O to the point T through P, Q, R and S. Taking A as the origin AB as the x-axis, AD as the y-axis and AH as the z-axis, determine:
   (a) the displacement vectors of the moves made by the cockroach.
   (b) the vector **OT**.
   (c) the position vector of the midpoint of OT.

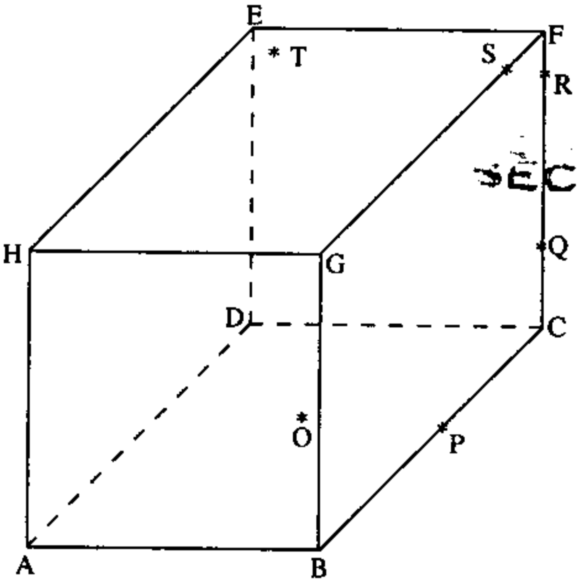


Fig. 11.3

7. A number of scouts climbed to the top of a hill from a base camp. Between the camp and the top of the hill, they rested at three different stations. The

first rest was 500 m higher than the base camp. It was 1 km to the east and 2 km to the north of the base camp. The second rest station was 250 m lower than the first rest station. It was 2 km to the west and 3 km to the north of the first station. The third rest station was 400 m higher than the second. It was 2 km west, and 1 km to the south of the second station. The top of the hill was 300 m higher than the third station. It was 500 m to the east and 2 km to the north of the third station. Taking the x-axis to be the east, the y-axis to be due north and the z-axis to be vertically upwards, write down the column vectors representing the course the scouts followed.

(a) How high was the top of the hill above the base camp?
(b) Describe the location of the top of the hill relative to the base camp.
(c) How far below the top of the hill was the second rest station?
(d) If a bird flew straight from the top of the hill to the base camp, express its course in terms of a single vector.

11.3: Column Vectors in terms of Unit Vectors i, j and k

In two dimensions the vector $\begin{pmatrix}1\\0\end{pmatrix}$ is the position Vector of a point on the x-axis one unit from the origin. It has unit magnitude and it is a unit vector in the direction of the x-axis. Similarly, write down the corresponding unit Vector in the direction of the y-axis.

In three dimensions, the unit vector in the direction of the x-axis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$, that in the direction of the y-axis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$ while that in the direction of z-axis is $\begin{pmatrix}0\\0\\1\end{pmatrix}$

These vectors are illustrated in figure 11.4 below.

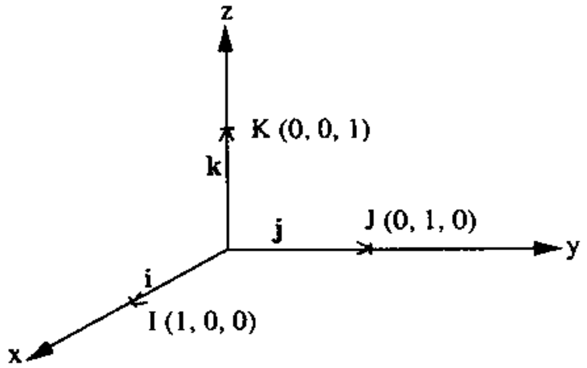


Fig. 11.4

Usually, the three unit vectors are written as; $\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Consider vector OE in figure 11.5. In this figure, OABCDEFG is a cuboid in which OA is 3 units, OC is 5 units and OG is 4 units.

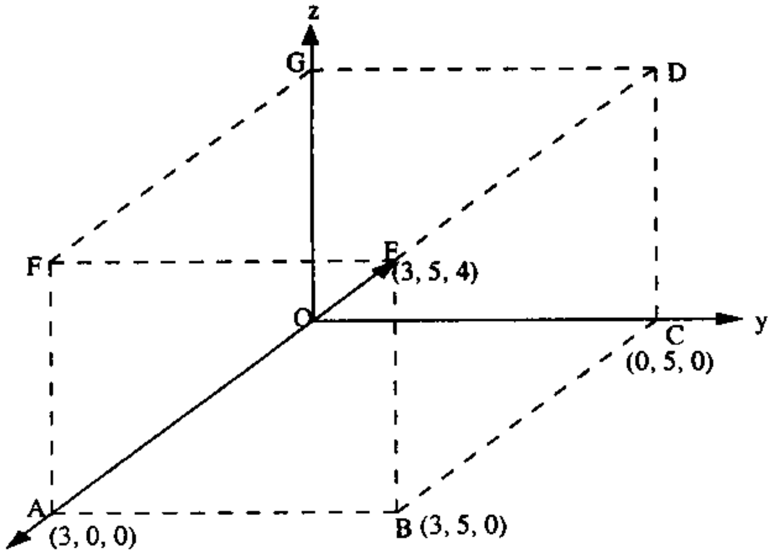


Fig. 11.5

The vector **OE** = $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

But **OE** = **OA** + **AB** + **BE**

$$= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

The vector OE has been expressed in terms of the unit vectors **i**, **j** and **k**.

Example 1

Express vector $\begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix}$ in terms of the unit vectors **i**, **j** and **k**.

Solution

$$\begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$$

$$= 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

In general, the column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be expressed as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Similarly, in two dimensions $\begin{pmatrix} a \\ b \end{pmatrix}$ can be expressed as $a\mathbf{i} + b\mathbf{j}$.

Example 2

If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$, find:

(a) $\mathbf{a} + \mathbf{b}$

(b) $3\mathbf{a} + 2\mathbf{b}$

Solution

(a)
$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + (3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}) \\ &= (2 + 3)\mathbf{i} + (1 - 4)\mathbf{j} + (5 + 7)\mathbf{k} \\ &= 5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k} \\ &= 12\mathbf{i} - 5\mathbf{j} + 29\mathbf{k}\end{aligned}$$

(a)
$$\begin{aligned}3\mathbf{a} + 2\mathbf{b} &= 3(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + 2(3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}) \\ &= 6\mathbf{i} + 3\mathbf{j} + 15\mathbf{k} + 6\mathbf{i} - 8\mathbf{j} + 14\mathbf{k} \\ &= (6 + 6)\mathbf{i} + (3 - 8)\mathbf{j} + (15 + 14)\mathbf{k} \\ &= 12\mathbf{i} - 5\mathbf{j} + 29\mathbf{k}\end{aligned}$$

Exercise 11.3

1. The position vectors of points A and B are **a** and **b** recpectively. Determine the column vector **AB**:
   (a) in two dimensions if:
      (i) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j}$
      (ii) $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$
      (iii) $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i}$
   (b) in three dimensions if:
      (i) $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$
      (ii) $\mathbf{a} = 5\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -3\mathbf{j} + 2\mathbf{k}$
      (iii) $\mathbf{a} = 4\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = -3\mathbf{j} + 2\mathbf{k}$
2. Given that $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{s} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{t} = 2\mathrm{i} + 3\mathrm{k}$ and 0 is the origin, determine the co-ordinates of P if **OP** is
   (a) $\mathbf{r} + \mathbf{s} + \mathbf{t}$ (b) $\mathbf{r} - \mathbf{s} + \mathbf{t}$
   (c) $\mathbf{r} - 3\mathbf{s} + \mathbf{t}$ (d) $\frac{1}{2}\mathbf{r} + \mathbf{s} - \frac{1}{2}\mathbf{t}$
3. Three points B, D and F are such that $\mathbf{BD} = 7\mathbf{i} + 14\mathbf{k}$, $\mathbf{BF} = -9\mathbf{j} + 14\mathbf{k}$. Given that L and M are midpoints of **BD** and **BF** respectively, find the column vectors **LM** and **DF**.
4. The position vectors of two points P and Q are $-\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. If R is the midpoint of **PQ**, determine the position vector of R.

11.4: Magnitude of a Vector in Three Dimensions

In figure 11.6 below, the vector **OE** is $\begin{pmatrix}4\\3\\5\end{pmatrix}$

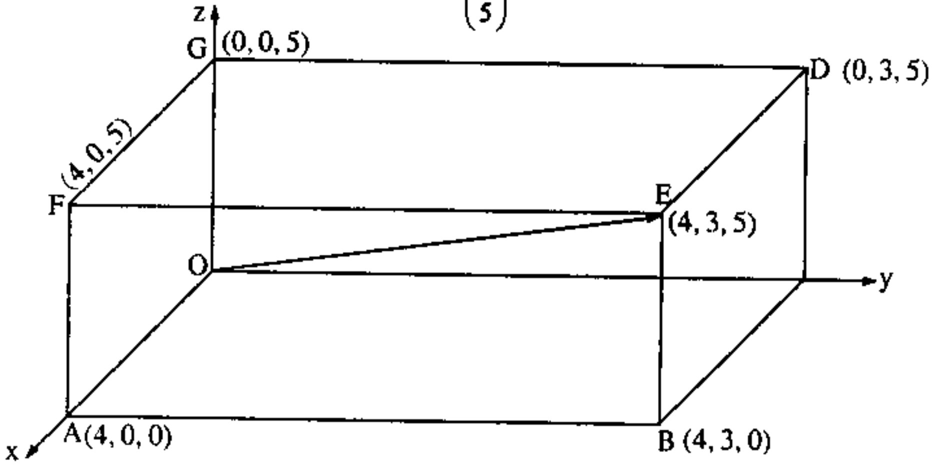


Fig. 11.6

The length of **OE** can be found by applying Pythagoras' theorem to the right- angled triangle OBE. Thus, **OE**^{2} = **OB**^{2} + **BE**^{2}

But OB^2 = OA^2 + AB^2

Therefore, **OE**^{2} = OA^2 + AB^2 + BE^2

$= 4^2 + 3^2 + 5^2$

OE $= \sqrt{4^2 + 3^2 + 5^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$ units

Similarly, we can determine the length of **BG**.

In the figure, **BG** = $\begin{pmatrix}-4\\-3\\5\end{pmatrix}$

Thus, the length of **BG** $= \sqrt{(-4)^2 + (-3)^2 + 5^2}$

$= \sqrt{16 + 9 + 25}$

$= 5\sqrt{2}$ units

Using the same figure, determine the length of **AD** and **CF**.

The length of a vector **AB** also called **mudulus** or **magnitude** of AB is written as |**AB**|

In general, given the vector **AB** = xi + yj + 2**k**, the n, |**AB**| = $\sqrt{x^2 + y^2 + z^2}$

Exercise 11.4

1. Find the modulus of each of the following vectors:

   (a) $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ (c) $\begin{pmatrix} -6 \\ 0 \\ 8 \end{pmatrix}$ (d) $\begin{pmatrix} -5 \\ -12 \\ 0 \end{pmatrix}$

2. The co-ordinates of A and B are (1, 6, 8) and (3, 0, 4) respectively. If O is the origin and **P** is the midpoint of AB, find the length of **OP**. How far apart are the midpoints of **OA** and **OB**?
3. Determine the magnitude of:

   (a) $-6\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ (b) $6\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$ (c) $-3\mathbf{i} - \frac{5}{2}\mathbf{j} + \frac{7}{2}\mathbf{k}$

   (d) $3\mathbf{i} + 4\mathbf{j}$ (e) $3\frac{3}{4}\mathbf{j} - 5\mathbf{k}$ (f) $7\mathbf{i} - 9\mathbf{k}$

   (g) $-14\mathbf{k}$ (h) $-7\mathbf{i} + 8\mathbf{i} + 14\mathbf{k}$
4. Given that $\mathbf{P} = 2i - j + 2k$ and $\mathbf{Q} = i + j - 2k$, determine;

   (a) $|\mathbf{p}|$ (b) $|\mathbf{q}|$ (c) $|\mathbf{p} + \mathbf{q}|$ (d) $|\mathbf{p} - \mathbf{q}|$

   (e) $|\mathbf{q} - \mathbf{p}|$ (f) $|2\mathbf{p} + 2\mathbf{q}|$ (g) $|2\mathbf{p} - 2\mathbf{q}|$ (h) $|\frac{1}{2}\mathbf{p} - 2\mathbf{q}|$
5. A room with a square floor ABCD of side 4 m is 3 m high. A moth is seen 1.5 m vertically above corner A. The moth then flies directly to a bulb located vertically above the centre of the floor and $\frac{1}{2}$ m below the ceiling. From the bulb, the moth then flies to corner C. Finally the moth crawls directly from C to point E on the ceiling and vertically above the midpoint of CD.
   (a) Write down the displacement vectors representing the paths followed by the moth.
   (b) What is the total distance covered by the moth.
   (c) How far from the starting point is the moth when it is at E?
6. In question 7 of exercise 11.2, determine the shortest distance of a straight line between:
   (a) the base camp and the first rest station.
   (b) the first rest station and the second rest station.
   (c) the second rest station and the top of the hill.
   (d) the base camp and the top of the hill.

11.5: Parallel Vectors and Collinearity

Parallel Vectors

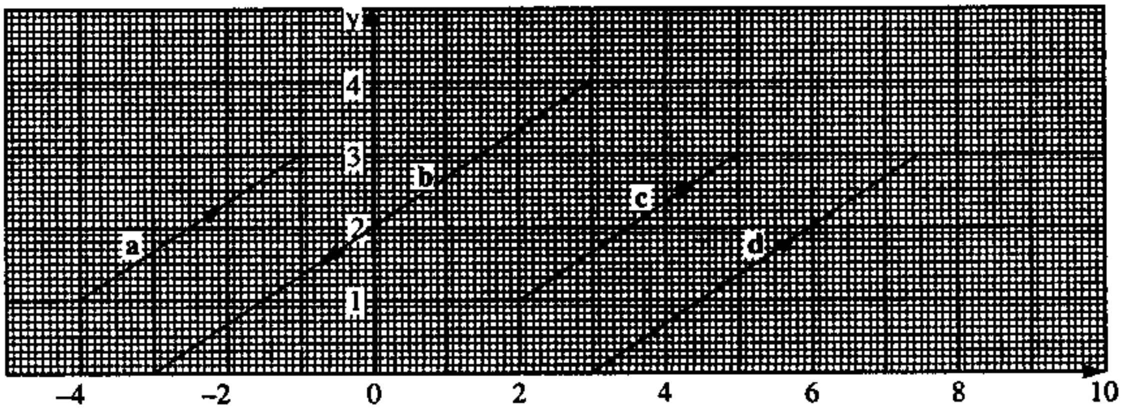


Fig. 11.7

Consider the parallel vectors **a**, **b**, **c** and **d** in figure 11.7. Expressing them as column vectors;

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} 4.5 \\ 3 \end{pmatrix}$

Expressing **b** in terms of **a**;

$\mathbf{b} = \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -2a$

Similarly;

$\mathbf{c} = \mathbf{a}$

$\mathbf{d} = 1.5\mathbf{a}$

Write **a**, **b** and **d** in terms of **c**.

If any two vectors are parallel, then one is a scalar multiple of the other. Conversely, if vector **a** is a scalar multiple of **b**, i.e., **a** = k**b**, then the two vectors are parallel.

Example 3

Which of the following pairs of vectors are parallel

(a) $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -12 \\ 9 \end{pmatrix}$ (b) $\mathbf{c} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 6 \\ -14 \end{pmatrix}$

Solution

(a) $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -12 \\ 9 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -12 \\ 9 \end{pmatrix} = -3\begin{pmatrix} 4 \\ -3 \end{pmatrix} = -3\mathbf{a}$

Therefore **a** and **b** are parallel.

(b) Vectors **c** and **d** are not parallel because neither is a scalar multiple of the other.

Determine which of the following pairs of vectors are parallel.

(i) $\mathbf{p} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 4.5 \\ -6 \end{pmatrix}$ (ii) $\mathbf{r} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} -6 \\ 4.5 \end{pmatrix}$

Collinear Points

Points are collinear if they lie on the same straight line. In this section we shall deal with the particular case of any three given points.

The test of collinearity, of three points consists of two parts:

(i) Showing that the column vectors between any two of them are parallel.

(j) Showing that they have a point in common.

Example 4

A (0, 3), B (1, 5) and C (4, 11) are three given points. Show that they are collinear.

Solution

AB and **BC** are parallel if **AB** = k**BC**, where k is a scalar.

$\mathbf{AB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\mathbf{BC} = \begin{pmatrix} 4 \\ 11 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Therefore, $\mathbf{AB} = \frac{1}{3}\mathbf{BC}$, or $\mathbf{BC} = 3\mathbf{AB}$

Thus, **AB//BC** and point B (1, 5) is common. Therefore, A, B and C are collinear.

Use **AC** and **AB** to establish the collinearity of the three points.

Example 5

Show that the points A(1, 3, 5), B (4, 12, 20) and C (3, 9, 15) are collinear.

Solution

Consider vector **AB** and **AC**.

$$\mathbf{AB} = \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 15 \end{pmatrix}$$

$$\mathbf{AC} = \begin{pmatrix} 3 \\ 9 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} = k\begin{pmatrix} 3 \\ 9 \\ 15 \end{pmatrix}$$

$$k = \frac{2}{3}$$

So, $\mathbf{AC} = \frac{2}{3}\mathbf{AB}$

Therefore, **AB//AC** and the two vectors share a common point A. The three points are thus collinear.

Example 6

In figure 11.8, **OA** = **a**, **OB** = **b** and **OC** = 3**OB**.

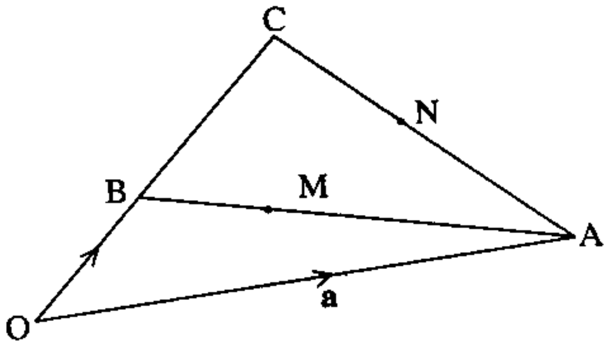


Fig. 11.8

(a) Express **AB** and **AC** in terms of **a** and **b**.

(b) Given that **AM** = $\frac{3}{4}$**AB** and **AN** = $\frac{1}{2}$**AC**, express **OM** and **ON** in terms of **a** and **b**.

(c) Hence, show that OM and N are collinear.

Solution

(a) $\mathbf{AB} = \mathbf{AO} + \mathbf{OB}$

$= -\mathbf{a} + \mathbf{b}$

$\mathbf{AC} = -\mathbf{a} + 3\mathbf{b}$

(b) $\mathbf{OM} = \mathbf{OA} + \mathbf{AM}$

$= \mathbf{OA} + \frac{3}{4}\mathbf{AB}$

$= \mathbf{a} + \frac{3}{4}(-\mathbf{a} + \mathbf{b})$

$= \mathbf{a} - \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

$= \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

$\mathbf{ON} = \mathbf{OA} + \mathbf{AN}$

$= \mathbf{OA} + \frac{1}{2}\mathbf{AC}$

$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + 3\mathbf{b})$

$= \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$

$= \frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$

(c) $\mathbf{OM} = k\mathbf{ON}$

$\frac{3}{4}\mathbf{b} + \frac{1}{4}\mathbf{a} = \frac{k}{2}\mathbf{a} + \frac{3k}{2}\mathbf{b}$

Comparing the coefficients of **a**;

$\frac{1}{4} = \frac{k}{2}$

$k = \frac{1}{2}$

Comparing coefficients of **b**;

$\frac{3}{4} = \frac{3}{2}k$

$k = \frac{1}{2}$

Thus, OM = $\frac{1}{2}$ON

The two vectors also share a common point, O. Hence, the points are collinear.

Exercise 11.5

1. Without drawing, state whether the following pairs of vectors are parallel or not:

   (a) $\begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}-6\\-3\end{pmatrix}$ (b) $\begin{pmatrix}-4\\4\end{pmatrix}, \begin{pmatrix}7\\-7\end{pmatrix}$ (c) $\begin{pmatrix}4\\-8\end{pmatrix}, \begin{pmatrix}-8\\4\end{pmatrix}$ (d) $\begin{pmatrix}0\\5\end{pmatrix}, \begin{pmatrix}0\\-20\end{pmatrix}$

   (e) $\begin{pmatrix}-4\\2\\1\end{pmatrix}, \begin{pmatrix}12\\-6\\-3\end{pmatrix}$ (f) $\begin{pmatrix}5\\3\\6\end{pmatrix}, \begin{pmatrix}25\\15\\30\end{pmatrix}$ (g) $\begin{pmatrix}7\\0\\2\end{pmatrix}, \begin{pmatrix}14\\-1\\4\end{pmatrix}$

   (h) $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
   $\mathbf{b} = 4.5\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$

   (i) $\mathbf{m} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\3\end{pmatrix}$ (j) $\mathbf{p} = \mathbf{r} - 2\mathbf{s}$
   $\mathbf{n} = 2\begin{pmatrix}4\\-1\end{pmatrix} - 2\begin{pmatrix}3\\4\end{pmatrix}$ $\mathbf{q} = 6\mathbf{s} - 3\mathbf{r}$

   (k) $\mathbf{r} = 2\mathbf{x} - 3\mathbf{y}$ (l) $\mathbf{m} = 7\mathbf{q} - 2\mathbf{p}$
   $\mathbf{s} = 3\mathbf{x} - 2\mathbf{y}$ $\mathbf{n} = \mathbf{p} - 3.5\mathbf{q}$

2. Use vector method to determine whether the following set of points are collinear. If they are, give the equation of the line:
   (a) (2, 4), (3, 7), (5, 13)
   (b) (3, 2), (7, 2), (8, 3)
   (c) (–5, 0), (–1, – 1), (5, –3)
   (d) (–3, –2), (–1, 0), (1, 2)
   (e) (0, –1), (2, 3), (5, 9)

3. Which of the following points are collinear?
   (a) (–1, 3, 4), (2, 5, 0), (–5, –1, –2)
   (b) (3, 4, 5), (9, 8, –10), (–15, –20, –25)
   (c) (–5, 2, 6), (–10, 4, 12), (–2.5, 1, 3)
   (d) (2, 3, 4), (1, –2, 0), (0, 2, –5)

(e) (4, 3, 2), (1, 0, 1), (7, 6, 3)

(f) (–1, –5, 6), (1, –2, 5), (5, 4, 3)

4. Given that the following pairs of vectors are parallel, determine the value of the unknown in each case:

(a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}a\\10.5\end{pmatrix}$ (b) $\begin{pmatrix}-4\\-3\end{pmatrix}, \begin{pmatrix}b\\1.5\end{pmatrix}$ (c) $\begin{pmatrix}0\\4\\-3\end{pmatrix}, \begin{pmatrix}o\\c\\15\end{pmatrix}$

(d) $\begin{pmatrix}d\\0.8\\10\end{pmatrix}, \begin{pmatrix}1.6\\0.32\\4\end{pmatrix}$ (e) $\begin{pmatrix}2.4\\1.6\end{pmatrix}, \begin{pmatrix}1.2\\e\end{pmatrix}$

5. In figure 11.9, OAB is a triangle. A is the point (2, 8) and B is the point (10, 2). C, D and E are the midpoints of OA, OB and AB recpectively.

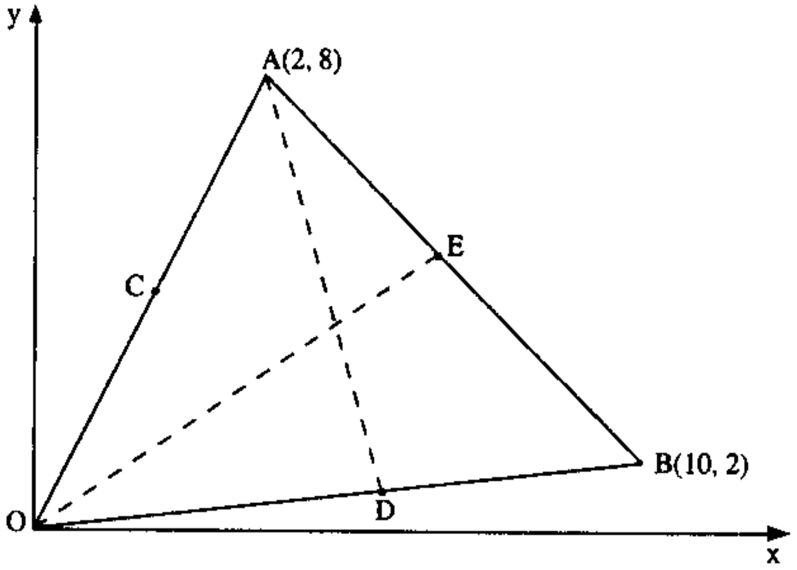


Fig. 11.9

(a) Find:
  (i) the co-ordinates of C and D.
  (ii) the lengths of the vectors CD and AB.

(b) Show that:
  (i) CD is parallel to AB.
  (ii) DE is parallel to OA.

6. Given that the points P (0, 3), Q (x, y) and R (4, 11) are collinear and that PQ : QR = 1 : 3, find the co-ordinates of Q.

7. Figure 11.10 shows a parallelogram ABCD. The midpoint of **AB**, **BC**, **CD** and **DA** are Q, R, S and T respectively. Use vectors to prove that QRST is also a parallelogram.

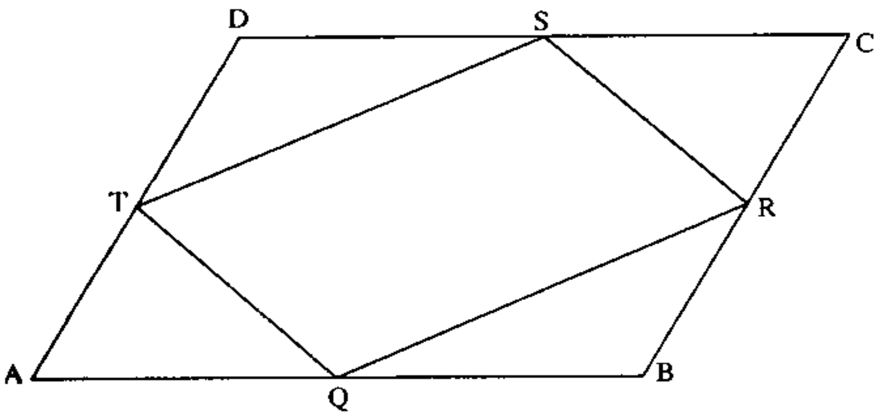


Fig. 11.10

8. In triangle OAB, **a** and **b** are the position vectors of A and B respectively. M is the midpoint of **AB**. If N and P are midpoints of **DB** and **OM** respectively, show that **PN//MB**.
9. Triangle OAB has vertices O(0, 0), A (6, 4) and B (8, 2). T, Q, R and S are midpoints of **OA**, **OB**, **AB** and **OR** respectively. Use vector method to show that T, Q and S are collinear.
10. Triangle OAB is such that OA = **a**, OB = **b**. C lies on **OB** such that **OC** : **CB** = 2 : 1. D lies on **AB** such that **AD** : **DB** = 1:2 and E lies on OA produced such that **OA** : **AE** = 3 : 1
    (a) Express the following vectors in terms of **a** and **b**:
        (i) **OC** (ii) **OD** (iii) **OE** (iv) **CD** (v) **DE**
    (b) Show that C, D and E are collinear.
11. In figure 11.11, the position vectors of a and b are **a** and **b** respectively. N is the midpoint of **AB**, **OL** : **LA** = 3 : 2 and **OM** : **MN** = 3 : 1. Express the vectors **ON**, **OM**, **LM** and **LB** in terms of **a** and **b**. Show that the points L, M and B are collinear. State the ratios **LM** : **MB**, **LM** : **LB**, **LB** : **MB**.

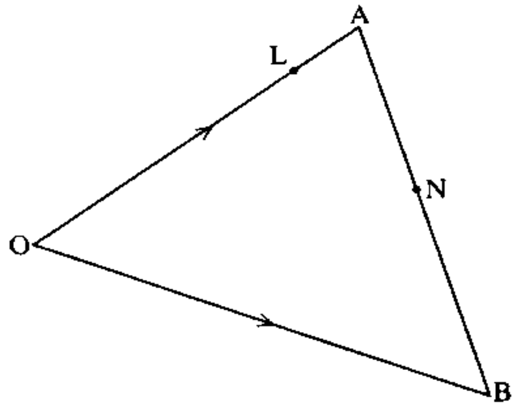


Fig. 11.11

12. In figure 11.12, OPRQ is a parallelogram, PS : SR = 2 : 1 and QR : RT = 2 : 1.

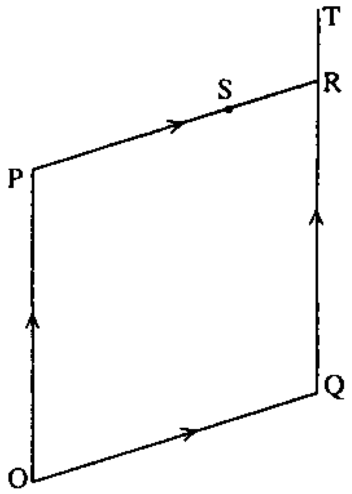


Fig. 11.12

Given that **OP** = **p**, **OQ** = **q** and **OR** = **r**, express in terms of **p** and **r** only:

(a) **QS**

(b) **OT**

Show that O, S and T are collinear.

13. Triangle AOB is such that **OA** = **a**, **OB** = **b**. C lies on **OB** such that **OC** : **CB** = 2 : 1. D lies on **AB** such that **AD** : **DB** = 1 : 2 and E lies on **OA** produced such that **OA** : **AE** = 3 : 1.
    (a) Express the following vectors in terms of **a** and **b**:
        (i) **OC** (ii) **OD** (iii) **OE** (iv) **DE**
    (b) Show that C, D and E are collinear.
14. OPQR is a trapezium in which **OP** = **p**, **OR** = **r** and **RQ** = 3**p**. **RQ** is produced to S such that **RQ** : **PS** = 3 : 1. T is a point on **PQ** such that **PQ** = 2 **PT**. Show that O, S and T are collinear.

11.6: Proportional Division of a Line

Internal division

In figure 11.12, the line is divided into 7 equal parts.

Fig. 11.12

The point R lies $\frac{4}{7}$ of the way along **PQ**. If we take the direction from P to Q to be positive, we say R divides **PQ** internally in the ratio 4 : 3. If instead we take as positive the direction from Q to P, we say that R divides **QP** internally in the ratio 3 : 4. In this case **QR** : **RP** = 3 : 4, or, 4**QR** = 3**RP**.

In figure 11.13, the line LM is divided into 15 equal parts.

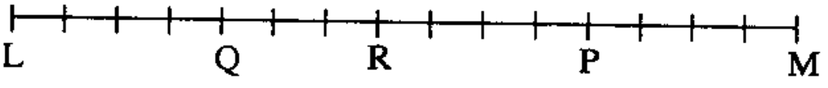


Fig. 11.13

In what ratio does:

(i) P divide **LM**.
(ii) R divide **ML**.
(iii) R divide **QP**.
(iv) Q divide **LR**.
(v) Q divide **PL**.

External Division

Note that in each of the above cases, we considered a point within a given interval. In the following section we consider points outside a given interval. Fig 11.14 shows a point P on **AB** produced.

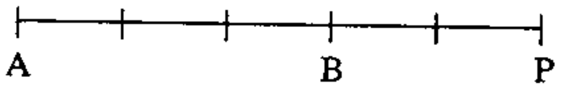


Fig. 11.14

The line **AB** is divided into three equal parts with **BP** equal to two of these parts. If we take the direction from A to B to be positive, then the direction from **P** to **B** is negative. Thus **AP** : **PB** = 5 : –2. In this case, we say that **P** divides AB externally in the ratio 5 : –2 or **P** divides AB in the ratio 5 : –2.

Using figure 11.13, find the ratio in which:

(i) R divides **LQ**
(ii) L divides **QR**
(iii) P divides **LR**
(iv) P divides **QR**
(v) R divides **PM**
(vi) Q divides **RP**

Exercise 11.6

1. In the figure 11.15, write down the ratios in which PS is divided by Q, R, T and U.

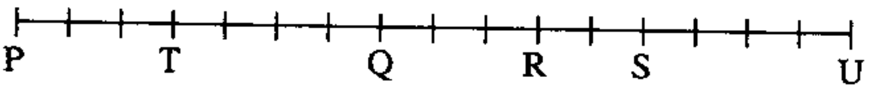


Fig. 11.15

2. In each of the following write down the value of AD : DE

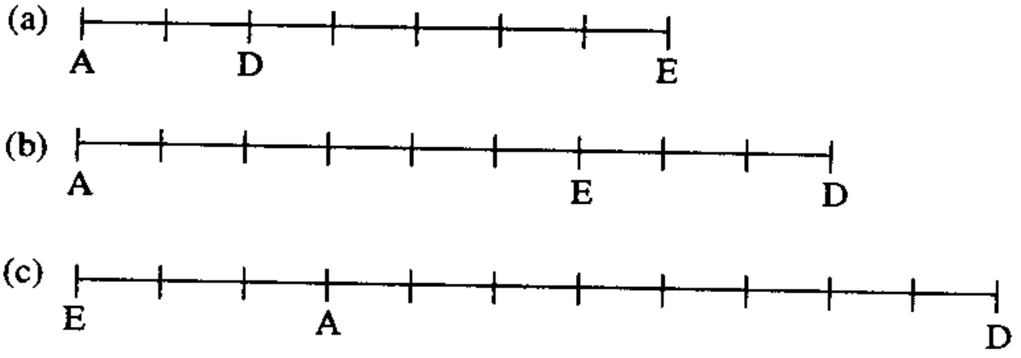


Fig. 11.6

3. Draw a line segment CD and show the position of X where **CX** : **XD** is
   (a) 2 : 5 (b) −2 : 5 (c) −2 : 1 (d) 3 : 1 (e) 6 : 5
4. In figure 11.17, **OC** = **c**. State the following vectors in terms of **c**.
   (a) **OA** (b) **OB** (c) **OD** (d) **DE**

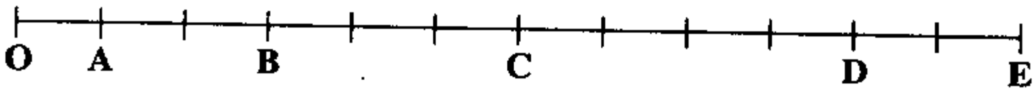


Fig. 11.17

5. A point P divides a line **AB** in the ratio 3 : 2 and a point Q divides AB externally in the ratio 7 : 2. Draw a sketch showing points A, B, P and Q. Hence, determine the ratio in which Q divides **PB**.
6. In figure 11.18, the line PQ is divided into 7 equal parts.

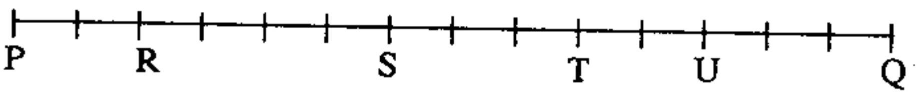


Fig. 11.17

Find the ratio in which:

(a) R divides **PS** (b) S divides **PR**
(c) S divides **TU** (d) T divides **SU**
(e) T divides **SQ** (f) T divides **UQ**
(g) U divides **RT**

7. A point Q divides a line **PR** internally in the ratio 2 : 1 and a point T divides the line internally in he ratio 3 : 1. In what ratio does T divide **PQ**?

11.7: The Ratio Theorem

In figure 11.19 below, R divides **AB** in the ratio 5 : 3.

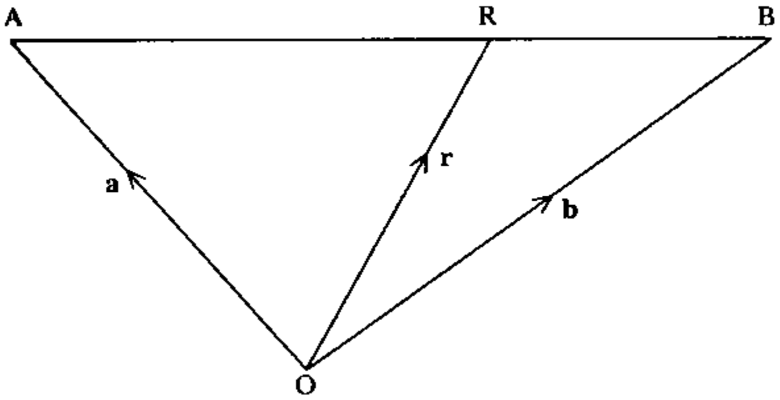


Fig. 11.19

We can express the position vector of R in terms of the position vectors of **a** and **b** with reference to any point O as origin. Let **OA** = **a**, **OB** = **b** and **OR** = **r**.
Then, **OR** = **OA** + **AR**

But **AR** = $\frac{5}{8}$**AB**

Therefore, **OR** = **OA** + $\frac{5}{8}$**AB**

$$\text{Thus } \mathbf{r} = \mathbf{a} + \tfrac{5}{8}(-\mathbf{a} + \mathbf{b})$$
$$= \mathbf{a} - \tfrac{5}{8}\mathbf{a} + \tfrac{5}{8}\mathbf{b}$$
$$= \tfrac{3}{8}\mathbf{a} + \tfrac{5}{8}\mathbf{b}$$

We can use the same method to find the position vector of a point which divides a given line externally. Consider figure 11.20, in which P divides **AB** externally in the ratio 5 : 1.

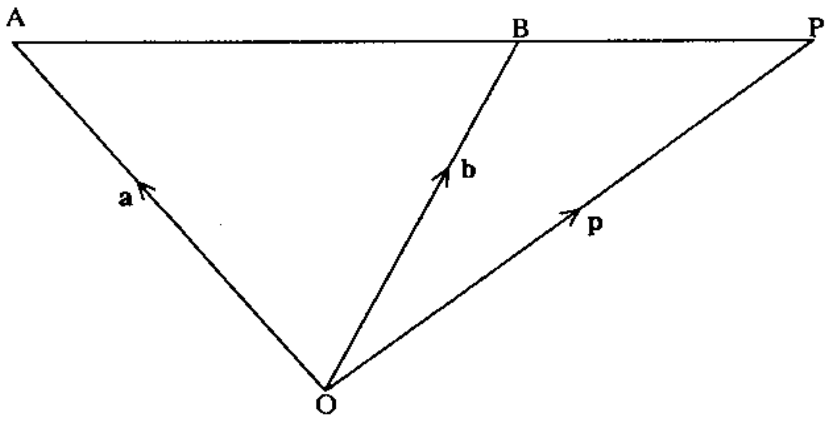


Fig. 11.20

Let **OA** = **a**, **OB** = **b** and **OP** = **p**

Then, **OP** = **OA** + **AP**

But **AP** = $\frac{5}{4}$**AB**

Therefore, **OP** = **OA** + $\frac{5}{4}$**AB**

$$\text{Thus, } \mathbf{p} = \mathbf{a} + \tfrac{5}{4}(-\mathbf{a} + \mathbf{b})$$

$$= -\tfrac{1}{4}\mathbf{a} + \tfrac{5}{4}\mathbf{b}$$

Let us now consider a general case. Figure 11.21, shows a point S which divides a line **AB** in the ratio m : n.

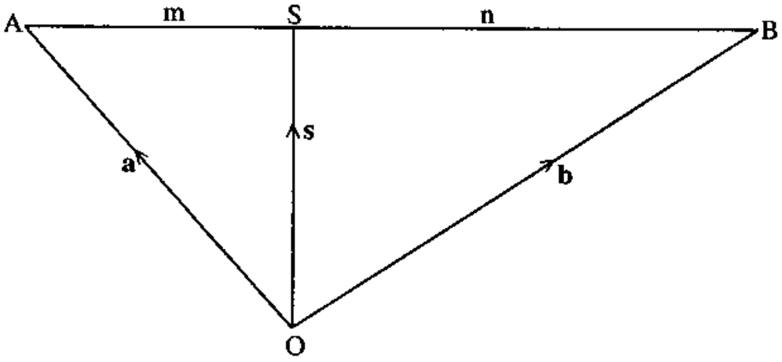


Fig. 11.21

Taking any point O as origin, we can express s in terms of **a** and **b** the position vectors of **a** and B respectively.

OS = **OA** + **AS**

But **AS** = $\frac{m}{m+n}$**AB**

Therefore, **OS** = **OA** + $\frac{m}{m+n}$**AB**

$$\text{Thus S} = \mathbf{a} + \frac{m}{m+n}(-\mathbf{a} + \mathbf{b})$$

$$= \mathbf{a} - \frac{m}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

$$= \left(1 - \frac{m}{m+n}\right)\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

$$= \left(\frac{m+n-m}{m+n}\right)\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

$$= \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

This is called the **ratio theorem**. The theorem states that the position vector **s** of a point which divides a line **AB** in the ratio m : n is given by the formula;

S = $\frac{n}{m+n}$ **a** + $\frac{m}{m+n}$**b**, where **a** and **b** are position vectors of A and B respectively. Note that the sum of co-ordinates $\frac{n}{m+n}$ and $\frac{m}{m+n}$ is 1.

Thus, in fig 11.19, the ratio m : n = 5 : 3
i.e., m = 5 and n = 3.

OR $= \frac{3}{5+3}\mathbf{a} + \frac{3}{5+3}\mathbf{b}$

Thus, $\mathbf{r} = \frac{3}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}$

Similarly in figure 11.20, the ratio m : n = 5 : –1

Therefore, P $= \frac{-1}{5+(-1)}\mathbf{a} + \frac{5}{5+(-1)}\mathbf{b}$

$= -\frac{1}{4}\mathbf{a} + \frac{5}{4}\mathbf{b}$

Example 7

A point R divides a line **QR** externally in the ratio 7 : 3. If **q** and **r** are position vectors of point Q and R respectively, find the position vector of **p** in terms of **q** and **r**.

Solution

We take any point O as the origin and join it to the points Q, R and P as shown in figure 11.22.

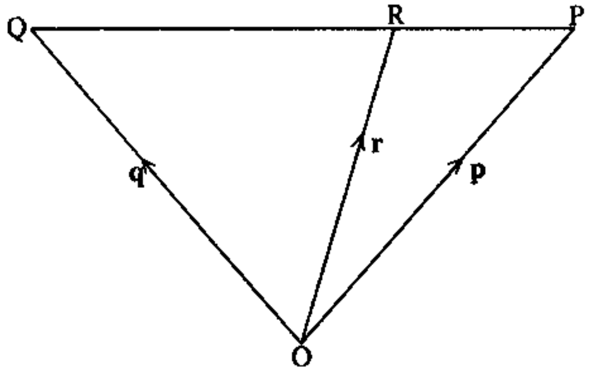


Fig. 11.22

QP : **PR** = 7 : –3

Substituting m = 7 and n = –3 in the general formular;

OP $= \frac{-3}{7+(-3)}\mathbf{q} + \frac{7}{7+(-3)}\mathbf{r}$

$\mathbf{p} = \frac{-3}{4}\mathbf{q} + \frac{7}{4}\mathbf{r}$

Vectors can be used to determine the ratio in which a point divides two lines if they intersect.

Example 8

In figure 11.23, OA = **a** and OB = **b**. A point P divides **OA** in the ratio 3 : 1 and another point Q divides **AB** in the ratio 2 : 5. If **OQ** meets **BP** at M, determine:

(a) **OM** : **MQ**

(b) **BM** : **MP**

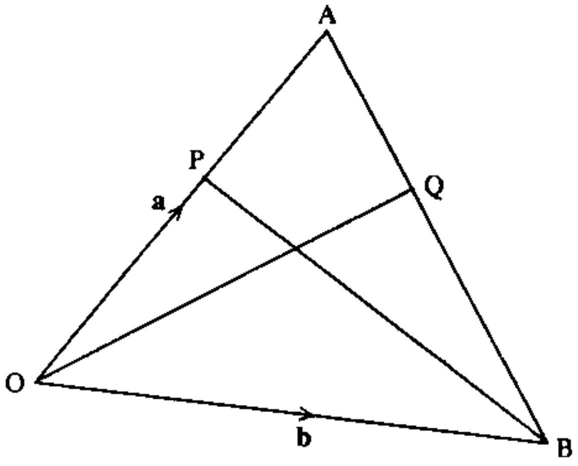


Fig. 11.23

Solution

Let **OM** : **MQ** = k : (1 – k)

and **BM** : **MP** = n : (1 – n)

Using the ratio theorem;

$$\mathbf{OQ} = \tfrac{5}{7}\mathbf{a} + \tfrac{2}{7}\mathbf{b}$$

$$\mathbf{OM} = k\mathbf{OQ}$$

$$= k(\tfrac{5}{7}\mathbf{a} + \tfrac{2}{7}\mathbf{b})$$

Also by ratio theorem;

$\mathbf{OM} = n\,\mathbf{OP} + (1 - n)\mathbf{OB}$

But $\mathbf{OP} = \tfrac{3}{4}\mathbf{a}$

Therefore, $\mathbf{OM} = n(\tfrac{3}{4}\mathbf{a}) + (1 - n)\,\mathbf{b}$

Equating the two expressions;

$k(\tfrac{5}{7}\mathbf{a} + \tfrac{2}{7}\mathbf{b}) = n\,(\tfrac{3}{4}\mathbf{a}) + (1 - n)\mathbf{b}$

Comparing coefficients;

$\tfrac{5}{7}k = \tfrac{3}{4}n$ .... (1)

$\tfrac{2}{7}k = 1 - n$ .... (2)

Solving simultaneously;

k $= \frac{21}{26}$ **and n** $= \frac{10}{13}$

Thus, OM : **MQ** $= \frac{21}{26} : \frac{5}{26} = 21 : 5$

The ratio BM : **MP** $= \frac{10}{13} : \frac{3}{13}$

$= 10 : 3$

Example 9

In figure 11.24, **OB** = **b** and **OA** = **a**. If **OA** : **AY** = 8 : 3 and **OB** : **BX** = 1 : 1, find **XD** : **DA** and **BD** : **DY**

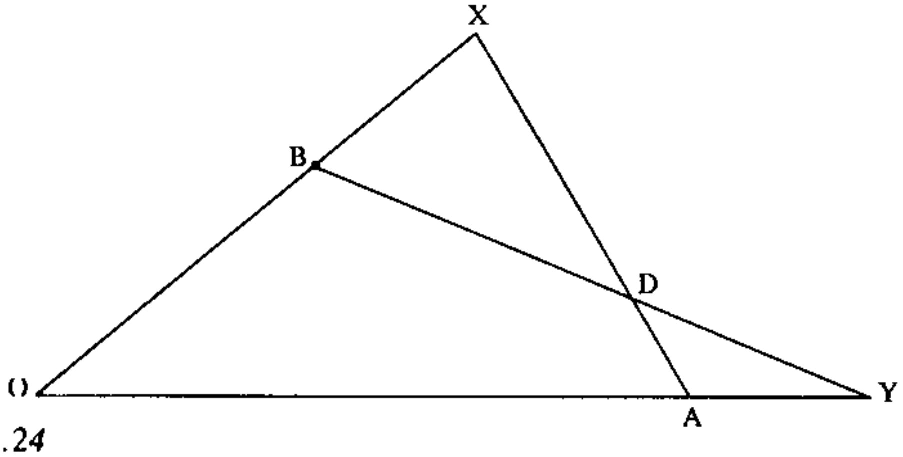


Fig. 11.24

Solution

Expressing **OD** in two ways;

Let **BD** = k**BY**

$$\mathbf{OD} = \mathbf{OB} + \mathbf{BD}$$

$$\mathbf{OD} = \mathbf{OB} + k\mathbf{BY}$$

$$= \mathbf{b} + k(\tfrac{8}{5}\mathbf{a} - \mathbf{b})$$

$$= \mathbf{b} - k\mathbf{b} + \tfrac{8}{5}k\mathbf{a}$$

$$= (1 - k)\,\mathbf{b} + \tfrac{8}{5}\,k\mathbf{a}$$

Similarly;

Let **XD** = t**XA**

$$\mathbf{OD} = \mathbf{oX} + t\mathbf{XA}$$

$$= 2\mathbf{b} + t\,(\mathbf{a} - 2\mathbf{b})$$

$$= 2\mathbf{b} + t\mathbf{a} - 2t\mathbf{b}$$

$$= 2\mathbf{b} - 2t\mathbf{b} + t\mathbf{a}$$

$$= (2 - 2t)\,\mathbf{b} + t\mathbf{a}$$

Comparing coefficients;

$1 - k = 2 - 2t$

$2t - k = 1$

$\frac{8}{5}k = t$

Solving the simultaneous equations;

$k = \frac{5}{11}, t = \frac{8}{11}$

Therefore, **BD** : **DY** $= \frac{5}{11} : \frac{6}{11}$

$= 5 : 6$

XD : **DA** $= \frac{8}{11} : \frac{3}{11}$

$= 8 : 3$

Exercise 11.7

1. Using figure 11.25, express the position vector of:
   - (a) T in terms of position vectors of S and U.
   - (b) p in terms of the position vectors of S and Q.
   - (c) Q in terms of the position vectors of P and U.
   - (d) S in terms of the position vectors of P and Q.

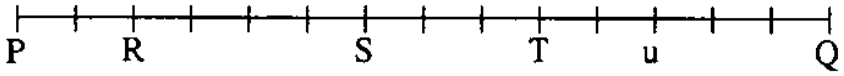


Fig 11.25

2. (a) The position vectors of F and G are $\begin{pmatrix}3\\6\end{pmatrix}$ and $\begin{pmatrix}9\\15\end{pmatrix}$ respectively. If a point M divides line **FG** in the ratio 1 : 2, find the position vector of M.
   (b) Repeat 2 (a) if the position vectors of F and G are $\begin{pmatrix}-1\\3\\4\end{pmatrix}$ and $\begin{pmatrix}5\\3\\-2\end{pmatrix}$ respectively.
3. (a) The co-ordinates of P are (0, 7) and Q are (3.5, 1.4). A point S divides **PQ** in the ratio 4 : 3. While a point T divides **PQ** externally in the ratio 9 : 2. Find co-ordinates of S and T.
   (b) Repeat 3 (a) if P and Q are the points (4, –1, 3) and (-3, 6, 3) respectively.
4. In figure 11.26, the position vectors of A, B and C are **a**, **b** and **c** respectively.

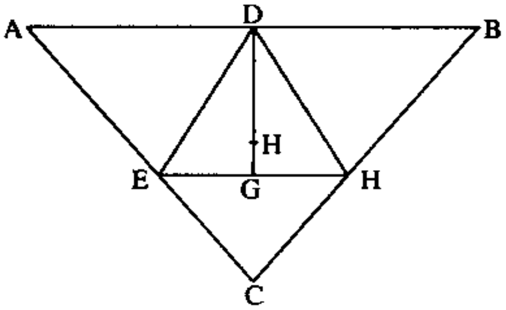


Fig. 11.26

Point D divides line **AB** in the ratio 1 : 1. Point E and F divide lines **AC** and **BC** in the ratio 2 : 1. Point G divides **EF** in the ratio 1 : 1 and H divides **DG** in the ratio 2 : 1. Find the position vector of H in terms of **a, b** and **c**.

5. Fig 11.27 shows a parallelogram ABCD in which **AB** and **AD** are subdivided into 9 and 6 equal intervals respectively.

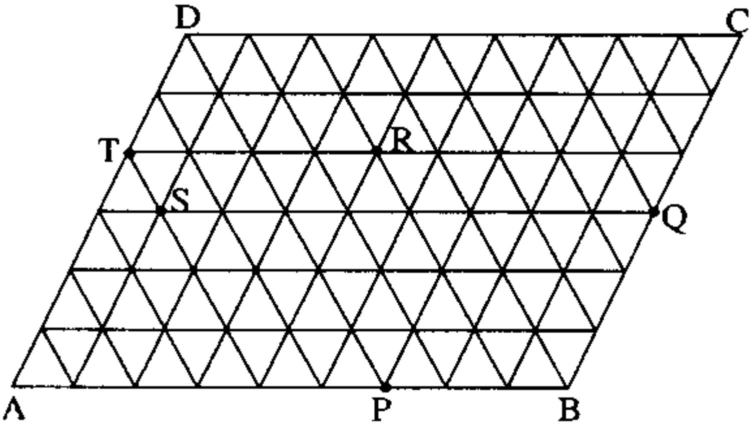


Fig. 11.27

The position vectors of A, B, C are **a, b** and **c** respectively.

(a) Express in terms of **a, b** and **c** the position vectors of:

(i) P	(ii) Q	(iii) D
(iv) T	(v) R	(vi) S

(b) Express in terms of **a, b** and **c** the position vector of the point of intersection of diagonals of the parallelogram ABCD.

6. Given that **a** = 8**p** + 6**q**, **b** = 10**p** – 2**q** and **c** = 2m**p** + 2 (m + n) **q**, where m and n are scalars, find the value of m and n if **c** = 6**a** – 4**b**.

7. Given that **OP** = 6**p** – 4**q**, **OQ** = 2**p** + 14**q** and **PQ** = 4 n**p** + (2n – m)**q** where n and m are scalars, find the values of n and m.

8. OABC is a trapezium with **OA** parallel to **CB** and **CB** = 4**OA**. D is a point on **OC** such that **OC** = 6 **OD**. **AD** and **OB** intersect at E. If **OA** = **a** and **OD** = **d**:

(a) express AD in terms of **a** and **d**.
(b) if **AE** = k**AD** where k is a scalar, express **OE** and **OB** in terms of **a** and **d**.
(c) if **OE** = h**OB** where h is a scalar, find the values of h and k. State the ratio **DE** : **EA**.

9. OAC is a triangle in which **OA** = **a** and **OC** = **c**. B divides **AC** in the ratio 3 : 2 and D divides **OC** in the ratio 1 : 2. **OB** meets **AD** at S. Express **OS** in terms of **a** and **c**.

10. OACB is a parallelogram in which **OA** = **a** and **OB** = **b**. Q divides **AC** in the ratio 5 : 3. **AB** meets **OQ** at P. Show that **OP** = $\frac{1}{13}$(8**a** + 5**b**).

11. In triangle ABC, **AB** = **c** and **BC** = **a**. Points M and N are on **BC** and **AC** respectively. If **BM** : **BC** = 1 : 4 and **AN** : **NC** = 3 : 2, express **AM** and **BN** in terms of **a** and **c**. If **AM** and **BN** meet at X, find the ratio **AX** : **XM**.

12. In a triangle PQR, N is a point on **QR** such that **QN** : **NR** = 5 : 2. PNMQ is a trapezium with **PN** parallel to **QM** and 5**PN** = 4 **QM**. If **PQ** = **q** and **PR** = **r**, express **QM** and **PM** in terms of **r** and **q**. If **PM** meets **QR** at L, find the ratios **PL** : **LM** and **QL** : **LR**.

13. In triangle ABC, D is a point on **AB** such that **AD** : **DB** = 1 : 2. E is a point on **AC** such that **AE** : **EC** = 2 : 3. **CD** and **BE** intersect at X. Find the ratios **CX** : **XD** and **BX** : **XE**.

14. In a triangle ABC, M is the midpoint of **BC**, N is a point on **AB** such that **AN** : **NB** = 1 : 2. If X is a point of intersection of **CN** and **AM**, determine the ratios **AX** : **XM** and **CX** : **XN**.

15. In a triangle ABC, M is a point on **BC** such that **BM** : **MC** = 3 : 2. A point N divides AB externally in the ratio 3 : 2. If **AB** = **a** and **AC** = **c**, find **AM**, **AN** and **MN** in terms of **a** and **c**. If **NM** produced meets **AC** at L, determine the ratios **AL** : **LC**.

16. The position vectors of points A and B with respect to an origin are **a** and **b** respectively. P is a point on **OA** such that **OA** = 3**OP**. Q is a point on **OB** such that **OQ** : **QB** = 2 : 1 and X is a point on PQ such that **PX** : **XQ** = 1 : 2. **OX** produced meets **AB** at R. Find **QX** and **QR** in terms of **a** and **b**.

17. In triangle PQR, M is the midpoint of **PR**. N is a point on **PQ** such that 3 **PN** = 2 **NQ**. **NM** produced meets **QR** produced at L. Determine the ratios in which L divides **QR**.

18. ABCD is a trapezium in which **AB** is parallel to **DC**. M and N are the midpoints of **AB** and **DC** respectively. The diagonals of AC and **BD** intersect at L and **DC** = 3**AD**.
(a) Show that triangles BLA and DLC are similar.
(b) Show that M, L and N are collinear.
(c) Determine the ratio **AL** : **LC**.

19. In a square PQRS, A is the midpoint of **PQ** and B is the midpoint of **QR**, while C is a point on **BP** such that 2**BC** = 3**CP**. The co-ordinates of P, Q and S are (4, 4), (14, 4) and (14, 14) respectively. Find:
  (a) (i) the column vectors of **PQ** and **PS**.
      (ii) co-ordinates of R.
  (b) Show that A, C, and S are collinear.
  (c) Determine the ratio in which C divides **AS**.

11.8: Applications of Vectors in Geometry

Vector method can be used to verify some well known geometrical theorems.

Example 10

Use vector method to show that the diagonals of a parallelogram bisect each other.

Solution

Figure 11.28 shows a parallelogram ABCD.

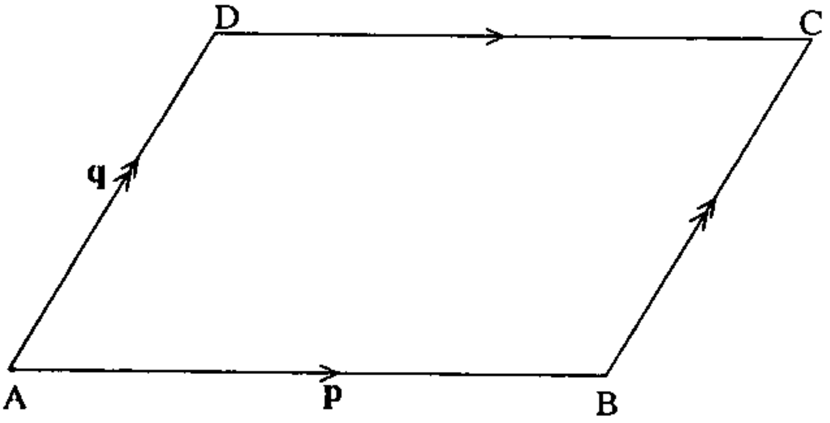


Fig 11.28

Taking A as the origin, **AB** = **p** and **AD** = **q**.
Let the two diagonals meet at X.
Expressing **AX** in two ways:

AX = k**AC**
= k(**AB** + **BC**)
= k (**p** + **q**)

AX = **AB** + **BX**
= **AB** + m**BD**
= **p** + M(− **p** + **q**)
= (1 − m) **p** + m**q**

Comparing coefficients:
k = 1 − m
k = m

Solving simultaneously;

$m = k = \frac{1}{2}$

$\mathbf{AX} = \frac{1}{2}\mathbf{AC} \Rightarrow \mathbf{AX} = \mathbf{XC}$. Also;

$\mathbf{BX} = \frac{1}{2}\mathbf{BD} \Rightarrow \mathbf{BX} = \mathbf{XD}$

Hence, the diagonals bisect each other.

Example 11

PQRS is a kite in which **PQ** = **PS** and **RQ** = **RS**. The midpoints of PQ, QR, RS and SP are A, B, C and D respectively. Use vector methods to show that ABCD is a rectangle.
Hint: The diagonals of a kite are perpendicular.

Solution

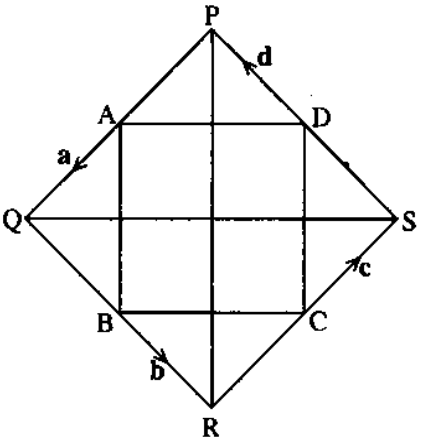


Fig. 11.29

Figure 11.29 shows the kite.
Let **PQ** = **a**, **QR** = **b**, **RS** = **c** and **SP** = **d**.
In the figure;

$\mathbf{DA} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$ and $\mathbf{SQ} = \mathbf{d} + \mathbf{a}$

Therefore, $\mathbf{DA} = \frac{1}{2}\mathbf{SQ}$

Hence, **DA** is parallel to **SQ**

Similarly, $\mathbf{BC} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{QS} = \mathbf{b} + \mathbf{c}$

Hence, $\mathbf{BC} = \frac{1}{2}\mathbf{QS}$

Therefore, **BC** // **QS**.
So, **AD** is parallel to **BC** (both are // to **QS**)
Similarly, **AB** is parallel to **DC** (both are // to **PR**)
But **PR** is perpendicular to **QS** (diagonals of a kite).

Hence AB and DC are both perpendicular to **QS** and therefore perpendicular to **BC and AD.**

Therefore ABCD is a parallelogram with each angle equal to 90°. Hence, ABCD **is a rectangle.**

Exercise 11.8

Use vector methods in this exercise

1. Prove that the diagonals of a rhombus bisect each other.
2. **ABCD** is a quadrilateral. M and N are the midpoints of **AB** and **DC** respectively. Show that **AD** + **BC** = **2 MN**
3. In a quadrilateral ABCD, K, L, M and N are the midpoints of **AB**, **BC**, **CD** and **DA** respectively. Prove that KLMN is a parallelogram.
4. In a triangle ABC, M and N are midpoints of **AB** and **AC**. Prove that **MN** is parallel to **BC** and **MN** = $\frac{1}{2}$**BC**.
5. If in a quadrilateral ABCD, **AB** is parallel and equal to **DC**. Prove that **AD** is parallel and equal to **BC**.
6. ABCD is a quadrilateral. M and N are the midpoints of BD and AC respectively. Show that **AB** + **AD** + **CB** + **CD** = 4 **MN**.
7. AM and BN are medians of a triangle ABC. G_1 is a point on **AM** such that **BG$_1$** : **GN** = 2 : 1. Find the position vector of G_1 with respect to an origin O outside the triangle. Similarly G_2 is a point on BN such that **BG$_2$** : **G$_2$N** = 2 : 1. Find the position vector of G_2 which divides the other median in the ratio 2 : 1. What do you deduce about the medians of a triangle?
8. In triangle ABC, M and N are points on **AB** and **AC** respectively such that **AM** : **MB** = **AN** : NC. Prove that **MN** is parallel to **BC**.
9. In figure 11.30, OP = **p** and **OQ** = **q**.

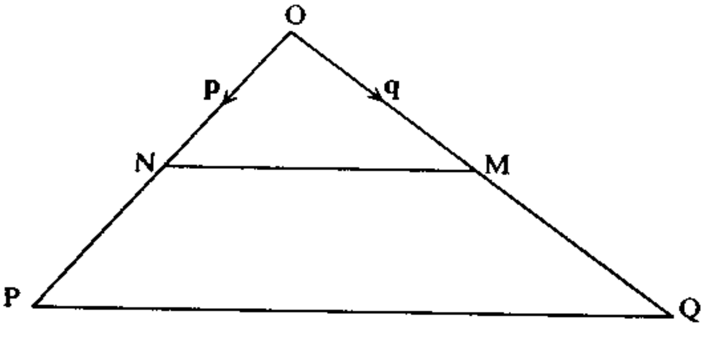
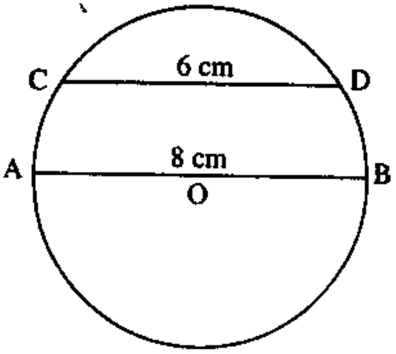


Fig. 11.30

If **ON** = $\frac{2}{3}$**OP** and **OM** = $\frac{2}{3}$**OQ**, show that **NM** is parallel to **PQ**.

Mixed Exercise 2

1. Given that $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 5 \\ 3 & 6 \end{pmatrix}$, find $\mathbf{A} + \mathbf{B}$.
2. If $\mathbf{P} = \begin{pmatrix} 1 & -7 \\ -3 & 2 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 6 & 9 \\ 3 & -7 \end{pmatrix}$, find:
   (a) $\mathbf{P} + \mathbf{Q}$ (b) $\mathbf{P} + \mathbf{Q} + \mathbf{R}$ (c) $\mathbf{R} + (\mathbf{P} + \mathbf{Q})$
   (d) $-\mathbf{P} + \mathbf{Q} + \mathbf{R}$ (e) $\mathbf{P} - \mathbf{Q}$
3. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, show that $\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{c}$.
4. Calculate the length of a chord of a circle of radius 13 cm which is 5 cm from the centre of the circle.
5. Find the values of the unknowns;
   $\begin{pmatrix} 2x & 3 \\ x & -3y \end{pmatrix} + \begin{pmatrix} -3 & 4 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -3 & 4 \end{pmatrix}$
6. The figure below is a circle with centre O and diameter AB is parallel to chord CD. Given that AB = 8 cm and chord CD is 6 cm long, calculate the distance of the chord from O.



7. Write down the fifth and the 11th terms of the following sequences:
   (a) 2, 15, 28, 41, ...
   (b) −2, 4, −8, 16, ...
   (c) 3 × 1, 4 × 2, 5 × 3, ...
   (d) $\frac{x+1}{3}, \frac{x+2}{5}, \frac{x+3}{7}, \ldots$
8. If P varies directly as Q and P = 5 when Q = 3, find Q when P = 15.
9. In each of the following find the determinant:
   (a) $\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{pmatrix}$
10. Find the distance between points $3\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

11. Make h the subject of the formula

$$E = 1 - \pi\sqrt{\frac{h - 0.5}{1 - h}}$$

12. Find the inverse of each of the following matrices:

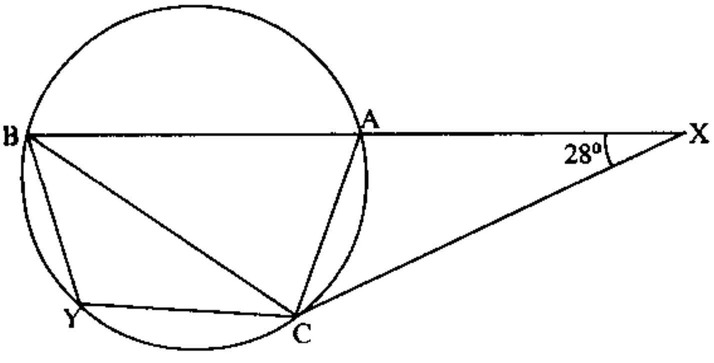
(a) $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (d) $\begin{pmatrix} x & -\frac{x}{2} \\ \frac{y}{2} & y \end{pmatrix}$

13. If an object is viewed through a transparent medium of thickness t cm and refractive index n, its apparent position is d cm nearer than its real one. The situation is represented by the formula; $d = t\frac{n-1}{n}$
Find n, the refractive index of the transparent medium, if d = 0.625 when t = 2.

14. X varies directly as the square root of Y and inversely as the cube root of Z. If X = 64 when Y = 4 and Z = 27, find X in terms of Y and Z.

15. The volume V of a gas varies directly as its temperature T and inversely as its pressure P. If T = 280 K when P = 70 cm and V = 190 cm^3, find V when T = 27 K and P = 78 cm.

16. In the figure below, XC is a tangent to the circle ABYC at C and Y is the midpoint of arc BC.
If $\angle BXC = 28^0$ and $\angle BCA = 2\angle ACX$, find $\angle CBA$, $\angle CBY$ and $\angle BYC$.



17. Given that **p** = **i** + **j**, **q** = 2**i** – **j** and **r** = 3**i**, find

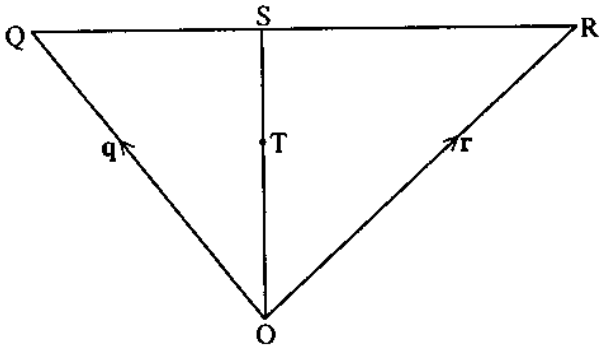
(a) **p** + **q** (b) 2**p** – **q** + **r** (c) $\frac{1}{2}$**p** + 2(**r** – **q**)

18. P is partly constant and partly varies as Q. If Q = 3 when P = 1 and Q = 4 when P = 10, find P in terms of Q. Also find P when Q = 6.

19. Write down the third and the fifth terms of a geometric series with the first term a and common ratio r. If the ratio of the third to the fifth term is

4, find the possible values of r. If a and r are both positive and the product of the two terms is 1, find the sum of the first eight terms of the sequence.

20. Q varies partly as the square of r and partly as the cube of r. When r = 10, Q = 208 and when r = 20, Q = 1.182. Find Q when r = 15.
21. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Find the value of r if $V = 311$ and $\pi = 3.142$.
22. Given that P varies directly as V and inversely as the cube of R and that P = 12 when V = 3 and R = 2, find V when P = 10 and R = 1.5.
23. Draw the graph of $y = 3x - 4x^2$ for values of x from $x = -5$ to $x = 5$ and use it to solve the following equations:
    (a) $3x - 4x^2 = -5$
    (b) $3x - 4x^2 = -25$
    (c) $3x - 4x^2 = x - 10$
    Try solving the equation $3x - 4x^2 = 3x + 5$. What do you notice?
24. In the figure below, S divides line **QR** in the ratio 1 : 2, T divides line **OS** in the ratio 3 : 2, **OR** = **r** and **OQ** = **q**.
    (a) Write in terms of **q** and **r** each of the following: **RQ**, **OS**, **RT**.
    (b) If L is the midpoint of line OQ, show that points R, T and L are collinear and find the ratio **RT** : **TL**.

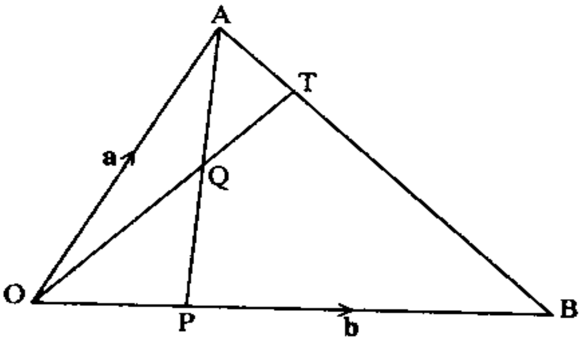


25. Given the sequence b, 3b, 5b, 7b, ..., determine:
    (a) the next three terms.
    (b) the 20th term.
    (c) the sum of the first 20 terms.
26. The 4th, 5th and 6th terms of a geometric sequence are $8k^3$, $16k^4$, $32k^5$ respectively. Determine:
    (a) the common ratio
    (b) the first three terms
    (c) the sum of the first 10 terms.
27. State which of the following sequences are geometric and those which are arithmetic. Find the sum of the first 20 terms where possible.

(a) 1, 2, 3, 4, 5, 6, ...
(b) b, b^2, b^3, ...
(c) 1, k + 2, 2k + 3, 3k + 4, ...
(d) 1, 2, 10, 12, 13, ...
(e) –2, 4, –8, 16, –32, ...

28. In a circle PQ is a diameter, SR is a chord parallel to PQ and T is a point on the minor arc SP. If ∠PQR = 58°, calculate ∠RPQ and ∠STP.

29. In the figure below, **OA** = **a** and **OB** = **b**.
Points P and T divide OB and AB internally in the ratio 2 : 3 and 1 : 3 respectively. Lines OT and AP meet at Q.



Find, in terms of **a** and **b**:
(a) **OT** (b) **OP** (c) **AP** (d) **OQ**

30. Given that A = $\begin{pmatrix} 2a & a \\ -3 & a \end{pmatrix}$ and that det A = 9, determine the value of a.

31. The tangent at the point C on a circle meets the diameter AB produced at T. if ∠BCT = 27°, calculate ∠CTA.

32. The first term of an arithmetic sequence is 2x + 1 and the common difference x + 1. If the product of the first and second terms is zero, find the first three terms of the two possible sequences.

33. OPQ is a triangle in which **OP** = **p** and **OQ** = **q**. R divides **PQ** in the ratio 2 : 3 and S divides **OQ** in the ratio 2 : 1. If **OR** meets **PS** at T express **OT** in terms of **p** and **q**. Hence, determine the ratio in which T divides **PS**.

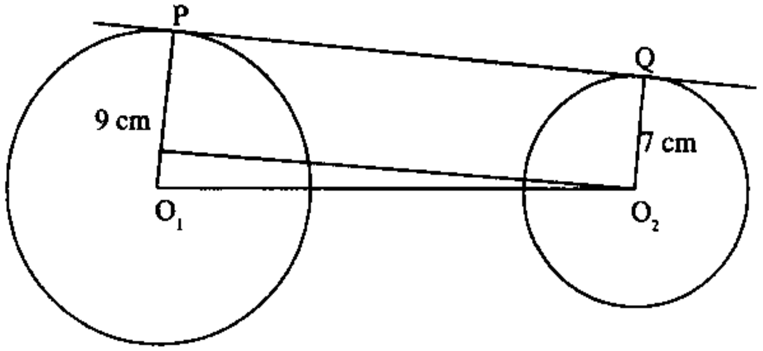
34. If A is (3, 6) and B is (8, –4), find the co-ordinates of the point X which divides **AB** internally in the ratio 3 : 2.

35. P, Q and R are three points on a circle such that ∠PQR = 120°. Find the angle between the tangents at P and R.

36. Given that **r** = 2**i** + 3**j** + 4**k**, **s** = 4**i** – **j** + 5**k** and **t** = –**i** + 9**j** + 5**k**, express **r** as a linear combination of s and **t**.

37. The figure below shows a pulley system with two wheels of radii 9 cm and 7 cm and centres O_1 and O_2 respectively. A continuous belt goes

round the wheels. Calculate the length of the belt if the distance between the centres of the wheels is 18 cm.



38. Two circles of radii 6 cm and 4 cm have their centre 6 cm apart. Construct a circle to touch the given circles internally. How many such circles can you construct?

39. If $\mathbf{P} = \begin{pmatrix} 6 & 1 \\ 2 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, find:

    (a) $\mathbf{PQ}$ (b) $\mathbf{P}^{-1}$ (c) $\mathbf{Q}^{-1}$ (d) $(\mathbf{PQ})^{-1}$ (e) $\mathbf{P}^{-1}\,\mathbf{Q}^{-1}$

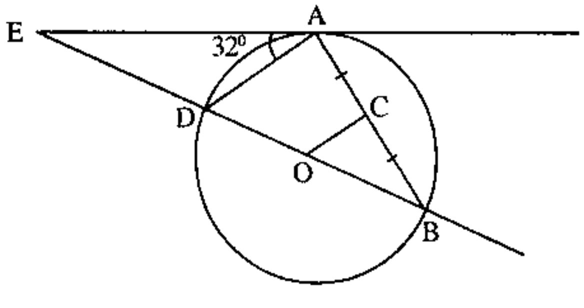
40. Find scalars λ and μ such that $\mu\mathbf{p} + \lambda\,\mathbf{q} = \mathbf{r}$, where $\mathbf{p} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$.

41. The points P(–2, 1), Q(1, 4) and R(3, 1) are vertices of a triangle PQR. If the triangle is given a translation **T** defined by the vector $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$, draw the triangle PQR and its image.

42. Show that if **r** and **q** are the position vectors of R and Q respectively, then the position vector of a point P on **RQ** such that RP : PQ = m : n, is $\frac{1}{m+n}(n\mathbf{r} + m\mathbf{q})$

43. A crossed belt passes over pulleys of diameter 18 cm and 12 cm. If the distance between the centres of the two pulleys is 60 cm, find the length of the belt.

44. In the figure below, O is the centre of the circle, AC = CB and AE is the tangent to the circle at A which meets BD produced at E.



If $\angle EAD = 32^0$, find $\angle BOC$ and $\angle AED$.

Chapter Twelve

BINOMIAL EXPANSIONS

A binomial is an expression of two terms. Some examples of binomials are $x + y$, $a + 3$, $2a + b$, and $2m^2 - n^3$.

12.1: Binomial Expansions up to Power Four

In book two, we saw the expansions such as;

$$(a + b)^2 = (a + b)(a + b)$$
$$= a^2 + 2ab + b^2$$

The idea can be used to expand higher powers of (a + b)

(i) $(a + b)^3 = (a + b)(a + b)(a + b)$

$$= (a + b)(a + b)^2$$
$$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

(ii) $(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$

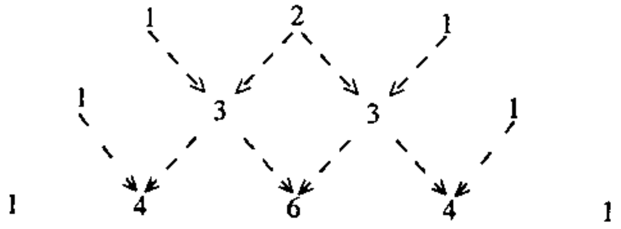
$$= (a + b)(a + b)^3$$
$$= (a + b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a(a^3 + 3a^2b + 3ab^2 + b^3) + b(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a^4 + 3a^3b + 3a^2b^2 + ab^3 + ba^3 + 3a^2b^2 + 3ab^3 + b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

12.2: Pascal's Triangle

The table below shows the expansions of $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$ together with their coefficients (as above).

Expression	*Expansion*	*Coefficients*
$(a + b)^2$	$a^2 + 2ab + b^2$	1 2 1
$(a + b)^3$	$a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a + b)^4$	$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1

The table of coefficients may be written as shown below:



Note:

(i) Each row starts with 1.

(ii) Each of the other numbers in the row is obtained by adding the two elements/numbers on either side of it in the preceding row as indicated by the arrows.

Write down the next three rows in the table. These are the coefficients of $(a + b)^5$, $(a +b)^6$ and $(a + b)^7$ respectively.

Recall that:

(i) $(a + b)^0 = 1$

(ii) $(a + b)^1 = (a + b)$

The table below shows the coefficients of the expansion of $(a + b)^n$ for $n = 0, 1, 2, \ldots 7$

$(a + b)^n$	*Coefficients*
$(a + b)^0$	1
$(a + b)^1$	1 1
$(a + b)^2$	1 2 1
$(a + b)^3$	1 3 3 1
$(a + b)^4$	1 4 6 4 1
$(a + b)^5$	1 5 10 10 5 1
$(a + b)^6$	1 6 15 20 15 6 1
$(a + b)^7$	1 7 21 35 35 21 7 1

The coefficients of the expansions form a pattern called Pascal's triangle. The pattern is named after the French Mathematician and philosopher, Blaise Pascal (1623-1662).

We can use Pascal's Triangle to obtain coefficients of expansions of the form $(a + b)^n$. For example, we have seen that the terms in the expansion of $(a + b)^4$ without coefficients are a^4, a^3b, a^2b^2, ab^3, b^4.

These can as well be written as a^4b^0, a^3b^1, a^2b^2, a^1b^3, a^0b^4.

Note:

(i) the sum of the powers of a and b in each term is equal to 4, which is the power to which $(a + b)$ is raised.

(ii) as powers of a descend, the powers of b ascend.

From Pascal's triangle, the coefficients of the above terms are

1 4 6 4 and 1 respectively.

Therefore;

$$\begin{aligned}(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Example 1

Expand $(p + q)^5$

Solution

In $(p + q)^5$, the terms without coefficients are:
$p^5, p^4q, p^3q^2, p^2q^3, pq^4$ and q^5.
From Pascal's triangle, the coefficients when n = 5 are;
1 5 10 10 5 1
Therefore, $(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

The extension of Pascal's triangle gives

$(a + b)^n$	*Coefficients*
$(a + b)^7$	1 7 21 35 35 21 7 1
$(a + b)^8$	1 8 28 56 70 56 28 8 1
$(a + b)^9$	1 9 36 84 126 126 84 36 9 1
$(a + b)^{10}$	1 10 45 120 210 252 210 120 45 10 1

Use Pascal's triangle above to expand:
(i) $(p + q)^8$ (ii) $(x + y)^9$ (iii) $(m + n)^{10}$

Example 2

Expand $(c + 4)^5$

Solution

In $(c + 4)^5$, terms without coefficients, are c^5, $c^4(4)$, $c^3(4)^2$, $c^2(4)^3$, $c(4)^4$, $(4)^5$
From Pascal's triangle, the coefficients when n = 5 are;
1 5 10 10 5 1

$$\text{Therefore, } (c + 4)^5 = c^5 + 5c^4 \times 4^1 + 10c^3 \times 4^2 + 10c^2 \times 4^3 + 5c \times 4^4 \times 1 \times 4^5$$
$$= c^5 + 20c^4 + 16c^3 + 640c^2 + 1\,280c + 1\,024$$

Example 3

Expand $(2x + 3y)^4$

Solution

$(2x + 3y)^4$
The terms without coefficients are;
$(2x)^4, (2x)^3(3y), (2x)^2(3y)^2, 2(x)(3y)^3, (3y)^4$
From Pascal's triangle, the coefficients when n = 4 are;
1 4 6 4 1
Therefore;

$$(2x + 3y)^4 = (2x)^4 + 4(2x)^3(3y) + 6(2x)^2(3y)^2 + 4(2x)(3y)^3 + (3y)^4$$
$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

Example 4

Expand $(x - y)^7$

Solution

$(x - y)^7 = (x + (-y))^7$

The terms without the coefficients are:

$x^7, x^6(-y), x^5(-y)^2, x^4(-y)^3, x^3(-y)^4, x^2(-y)^5, x(-y)^6, (-y)^7$

From Pascal's triangle, the coefficients when n = 7 are;

1 7 21 35 35 21 7 1

$\therefore\ (x - y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

Exercise 12.1

1. Expand:
   (a) $(x + y)^4$ (b) $(a + b)^8$ (c) $(p + q)^7$ (d) $(y + z)^{10}$
2. Expand the following:
   (a) $(x + 4y)^3$ (b) $(1 + x)^6$ (c) $(2x + 1)^4$ (d) $(2z + 3y)^5$
3. Expand:
   (a) $(x - y)^5$ (b) $(b - c)^6$ (c) $(m - n)^9$ (d) $(t - k)^{10}$
4. Expand:
   (a) $(x - 3y)^5$ (b) $(2x - 3)^4$ (c) $(-3 - x)^6$
   (d) $(b - 3c)^5$ (e) $(1 - p)^4$
5. Expand each of the following:
   (a) $(x + \frac{1}{2})^3$ (b) $(p - \frac{1}{4}q)^4$ (c) $(q - \frac{1}{3}p)^3$
   (d) $(y - \frac{1}{y})^5$ (e) $(3x + \frac{1}{3})^4$ (f) $(x - \frac{1}{x})^8$
   (g) $\frac{r}{3} + \frac{3}{r})^6$
6. Expand the following, giving your answers in surds:
   (a) $(1 + \sqrt{5})^3$ (b) $(1 - \sqrt{5})^6$ (c) $(1 + 2\sqrt{3})^4$
   (d) $(2 - 3\sqrt{7})^5$ (e) $(3 - \frac{1}{2}\sqrt{2})^4$ (f) $(2\sqrt{3} + 3\sqrt{2})^6$
7. By writing $1 + x + x^2$ as $1 + (x + x)^2$, use the binomial expansion to expand:
   (a) $(1 + x + x^2)^2$ (b) $(1 - x - x^2)^3$
8. Work out $(2 - \sqrt{x})^4 + (2 + \sqrt{x})^4$
9. Find the constant term in the expansion $(2y + \frac{1}{y})^4$

12.3: Applications to Numerical Cases

Binomial expansion can be used to find solutions to numerical problems.

Example 5

Use binomial expansion to evaluate $(1.02)^6$ to 4 s.f.

Solution

$(1.02) = (1 + 0.02)$

$\therefore (1.02)^6 = (1 + 0.02)^6$

The terms without coefficients are

$1^6, 1^5(0.02)^1, 1^4(0.02)^2, 1^3(0.02)^3, 1^2(0.02)^4, 1^1(0.02)^5, (0.02)^6$

From Pascal's triangle, the coefficients when n = 6 are;

1 6 15 20 15 6 1

Therefore;

$$\begin{aligned}(1.02)^6 &= 1 + 6(0.02) + 15(0.02)^2 + 20(0.02)^3 + 15(0.02)^4 + 6(0.02)^5 + (0.02)^6\\ &= 1 + 0.12 + 0.0060 + 0.00016 + 0.0000024 + 0.0000000192\\ &\quad + 0.000000000064\\ &= 1.1261624\\ &= 1.126 \text{ (4 s.f.)}\end{aligned}$$

Note:

To get the answer, it is sufficient to consider the addition up to the 4th term of the expansion. Subsequent terms are too small to affect the answer.

Example 6

Expand $(1 + x)^9$ up to the term in x^3. Use the expansion to estimate $(0.98)^9$ correct to 3 decimal places.

Solution

$(1 + x)^9$

The terms without the coefficients are;

$1^9, 1^8(x), 1^7(x)^2, 1^6(x)^3, 1^5(x)^4. \ldots$

From Pascal's triangle, the coefficients when n = 9 are:

1, 9, 36, 84, 126, 126, 84, 36, 9, 1

$\therefore (1 + x)^9 = 1 + 9x + 36x^2 + 84x^3 + \ldots$

$(0.98)^9 = (1 - 0.02)^9$

Here, $x = -0.02$

$$\begin{aligned}\therefore (0.98)^9 &= 1 + 9 \times (-0.02) + 36 \times (-0.02)^2 + 84\,(-0.02)^3\\ &= 1 - 0.18 + 0.0144 - 0.000672\\ &= 0.833728\\ &= 0.834 \text{ (3 decimal places)}\end{aligned}$$

Example 7

Expand $(1 + \frac{1}{2}x)^{10}$ up to the term in x^3 in ascending powers of x. Hence, find the value of $(0.005)^{10}$ correct to four decimal places.

Solution

Using the 11th row of Pascal's triangle;

$(1 + \frac{1}{2}x)^{10} = 1 + 10(\frac{1}{2}x) + 45(\frac{1}{2}x)^2 + 120(\frac{1}{2}x)^3$

$= 1 + 10 \times \frac{1}{2}x + 45 \times \frac{1}{4}x^2 + 120 \times \frac{1}{8}x^3$

$= 1 + 5x + \frac{45}{4}x^2 + 15x^3$

$(1.005)^{10} = (1 + 0.005)^{10}$

Here, $\frac{1}{2}x = 0.005$

$\therefore x = 0.010$

Substituting for x = 0.01, in the expansion;

$\{1 + \frac{1}{2}(0.01)\}^{10} = 1 + 5 \times 0.01 + \frac{45}{4} \times (0.01)^2 + 15(0.01)^3$

$= 1 + 0.05 + 0.001125 + 0.000015$

$= 1.051140$

$= 1.0511$ (4 decimal places)

Exercise 12.2

1. Use binomial expansion to evaluate each of the following, correct to 4 decimal places:
   (a) $(1.02)^5$ (b) $(1.003)^3$
   (c) $(0.97)^4$ {Hint: 0.97 = 1 – 0.03}
   (d) $(0.98)^3$ (e) $(1.99)^5$ (f) $(0.96)^4$
2. Use binomial expansion up to the 4th term to estimate each of the following to 4 decimal places:
   (a) $(1.005)^{10}$ (b) $(1.01)^9$ (c) $(0.99)^8$ (d) $(0.999)^6$.
3. Use binomial expansion to evaluate the following correct to 4 s.f.:
   (a) $(5 + 0.05)^4$ (b) $(3 - 0.03)^4$ (c) $(1.98)^6$
4. Write down the expansion of $(1 + \frac{1}{4}x)^4$. Taking the first three terms of the expansion and putting x = 0.1, find the value of $(1.025)^4$ correct to three decimal places.
5. Use Pascal's triangle to expand $(1 + \frac{2}{x})^4$. Use your expansion to find the value of $(1.02)^4$ correct to four significant figures.
6. Expand $(a - b)^5$. Hence find the value of $(0.96)^5$, correct to 3 decimal places.
7. Write the expansion of $(2 - \frac{1}{5}x)^5$. Hence, use the expansion to find the value of $(1.96)^5$ correct to 3 decimal places.
8. Expand $(2 + 3x)^6$ up to the term in x^4. Hence, estimate $(1.91)^6$ correct to 4 decimal places.

Chapter Thirteen

PROBABILITY

1.1: Introduction

Consider the following situations:

(i) A candidate in a certain school would like to know his/her chances of passing the final examination.

(ii) A farmer would like to know the likelihood of his cow giving birth to a heifer if for the last six births it has given birth to five bulls and one heifer.

(iii) A and B decide to play a game of tossing a coin. They agree that A wins when a head shows up and B wins when a tail shows up. Each player would like to know his/her chances of winning the game.

(iv) At the start of a football match, a coin is tossed to decide which team takes the northern side of the field. Each team would like to know its chances of starting on their favoured side.

(v) A traveller would like to know which is safer to travel by road or by rail.

The situations above are examples in daily life where we are interested in telling the chance or likelihood of an event occurring. In other words, we are trying to tell how probable it is that an event will occur.

In situation (ii) the chances of the farmer's cow giving birth to a heifer can be determined from past experience. We notice that in the last six births, the cow got one heifer and five bulls. Assuming the same trend, we may say that the chances of the cow giving birth to a heifer is one out of six and that of giving birth to a bull is five out of six.

This numerical measure of the chance is called **probability**. In this case, we say that the probability that the cow gives birth to a heifer is $\frac{1}{6}$and that of getting a bull is $\frac{5}{6}$.

13.2: Experimental Probability

A fair coin was tossed 100 times and the number of heads and tails showing up were recorded as 54 and 46 respectively. A student wants to determine the probability of obtaining a head with this coin when she/he tosses it one more time.

Out of the 100 tosses, 54 were heads. The likelihood of she/he obtaining a head this one more time is $\frac{54}{100} = \frac{27}{50}$. The probability of a head occurring may be denoted by P(Head) or P(H) $\frac{54}{100} = \frac{27}{50}$

In the above situation, the probability has been determined from experience or experiment. This is what is referred to as experimental probability.

Note:

(a) What is done or observed together with the recorded observation is the experiment.

(b) Each toss is called a trial and in this case there are 100 trials.

(c) The possible result of a trial is called an outcome.

(d) Generally experimental probability of a result is given by

$$\frac{\text{the number of favourable outcomes}}{\text{the total number of trials}}$$

Example 1

A boy had a fair die with faces marked 1 to 6. He threw this die up 50 times and each time he recorded the number on the top face. The result of his experiment is shown in the table below.

Table 13.1

Face	1	2	3	4	5	6
Number of times a Face has shown up	11	6	7	9	9	8

What is the experimental probability of getting:

(a) 1 (b) 4 (c) 6?

Solution

(a) $\text{P(Event)} = \frac{\text{the number of favourable outcomes}}{\text{the total number of trials}}$

$P(1) = \frac{11}{50}$

(b) $P(4) = \frac{9}{50}$

(c) $P(6) = \frac{8}{10} - \frac{4}{25}$

Example 2

From the past records, out of the ten matches a school football team has played, it has won seven. How many possible games might the school win in thirty matches?

Solution

P (Winning in one match) = $\frac{7}{10}$

∴ **The number** of possible wins in thirty matches $= \frac{7}{10} \times 30$

$= 21$ matches

Exercise 13.1

1. Toss a coin 100 times and record the number of times a head shows up. Use the result of your experiment to determine the probability of a head showing up in a toss.
2. Draw four parallel lines whose distances apart are twice the length of a paper pin. Throw the paper pin up 100 times so as to land on the paper containing the lines. Record the number of times the pin rests on the paper without touching any line. What is the experimental probability that the pin does not touch any line?
3. Statistics have shown that 38 out of every 1 000 new bulbs of a certain type burn out within 6 months. What is the probability of buying a bulb which will not burn out in 6 months time?
4. Throw two dice, whose faces are numbered 1 to 6, fifty times. Record the sum of the numbers that appear on their tops each time. Find the experimental probability that the sum is:
   (a) 7 (b) 1 (c) 2 (d) 4 (e) 13
5. After tossing a coin 10 times, it was found that the probability of getting a head is 0.4. What is the probability of getting a tail in this experiment?
6. Out of 20 days, Rono did not use his pen for 4 days because he did not fill it with ink and he did not use it for 3 days because he forgot it at home. Find the probability that in a given day during that period;
   (a) he forgot his pen at home,
   (b) he used his pen in school.
7. Open any page of your English textbook at random. Count the number of words on the page. Count also the number of four-letter words. Assuming that the distribution of words throughout the book is the same, what is the probability that a word in the book has four letters.
8. The number of days with rainfall at a station was recorded for the whole year in a table as shown below.

Table 13.2

Month	J	F	M	A	M	J	J	A	S	O	N	D
No. of days with rainfall	3	4	8	18	9	5	4	6	9	12	6	4

Use the information to find:
(a) the probability that a day in November had some rain.
(b) the number of months for which the probability of a day with rain is less than 0.3.
(c) the probability that a day in the last three months of the year has some rain.
(d) the probability that a day in the year will have rain.

9. The statistics club of a school recorded the number of persons in each passing vehicle along a road for 1 hour as follows:

Table 13.3

No. of persons in vehicle	1	2	3	4	5
No. of vehicles	60	50	20	30	10

Find the probability that a vehicle passing along the road had:
(a) an odd number of persons.
(b) an even number of persons.

10. Form three students in a certain school recorded at ten minute intervals the number of vehicles passing at a point on the road near their school and noted the number of *matatus*. Table 13.4 below shows their results.

Table 13.4

Time	*Total no. of vehicles*	*No. of "matatus"*
10.00 – 10.10	9	3
10.10 – 10.20	13	4
10.20 – 10.30	7	2
10.30 – 10.40	10	3
10.40 – 10.50	7	1
10.50 – 11.00	12	5

(a) How many vehicles passed during the observation period?
(b) How many *matatus* passed?
(c) What is the probability that a *matatu* will pass at any one time between 10.00 a.m. and 11.00 a.m.?

13.3: Range of Probability Measure

Consider the following example:

If today is a Monday, certainly the day tomorrow will be Tuesday. It is impossible for the day tomorrow to be any other day of the week apart from Tuesday so long as today is a Monday.

Now that we are certain that the day tomorrow will be a Tuesday, then the probability of the day tomorrow to be a Tuesday is 1 while the probability of the day tomorrow being any other day of the week apart from Tuesday is 0.

From this example and the results you obtained by working problems in exercise 13.1 you notice the following:

(a) The greater the probability the more the event is likely to take place.

(b) The probability of an event A which is certain to occur is 1, i.e., $P(A) = 1$

(c) The probability of an event A which is impossible to occur is 0, i.e., $P(A) = 0$.

(d) The probability of any event A lies between O and 1 (inclusive). That is for any event A, $O \le P(A) \le 1$. This is the range of probability measure.

(e) If $P(A)$ is the probability of an event A happening and $P(A')$ is the probability of an event A not happening, then $P(A') = 1 - P(A)$ and $P(A') + P(A) = 1$

Probabilities are expressed as fractions, decimals or percentages.

13.4: Probability Space

A fair coin is tossed once. We have already mentioned that the only possible outcomes from such a single toss of a coin are the occurrence of a head or the occurrence of a tail, and vice versa. These outcomes can be listed as H, T. This list of all possible outcomes is called a **possibility space** or a **sample space** or a **probability space**.

The physical nature of the coin is such that the head or tail have equal chances of occurring. The events head or tail are said to be equally likely or equiprobable.

13.5: Theoretical Probability

If two coins are tossed together once, the probability space is HH, TT, TH, HT. This list may be displayed as shown in the outcome table 13.5 below.

Table 13.5

		First coin	
Second coin	▨	H	T
	H	HH	HT
	T	TH	TT

In this case, there are four possible outcomes. Thus;

(i) P (two heads) = $\frac{1}{4}$

(ii) P (a head and a tail) = $\frac{2}{4} = \frac{1}{2}$

(iii) P (two tails) = $\frac{1}{4}$

Note that the probability of the above events can be calculated without necessarily using any past experience or doing any experiment. This kind of probability is then referred to as **theoretical probability**.

Tossing a Fair Die

(i) If a fair die is tossed once, what is the probability of a 3 showing up? Since the die has six faces and each face is equally likely to come up, the probability space is 1, 2, 3, 4, 5 and 6

$$\text{Therefore, } P(3) = \frac{\text{number of times a 3 show up}}{\text{number of possible outcomes}}$$

$$= \frac{1}{6}$$

This is because a 3 appears only once out of the six possible outcomes.

(ii) What is the probability of getting an odd number when a fair die is tossed once? Out of the six possible outcomes, there are three odd numbers; 1, 3 and 5.

$$\text{Therefore, } P(\text{odd}) = \frac{3}{6}$$

$$= \frac{1}{2}$$

(iii) What is the probability of getting a prime number on a single toss of a fair die?

There are prime numbers: 2, 3 and 5. Therefore P (prime) = $\frac{3}{6}$

$$= \frac{1}{2}$$

When a fair die is tossed, the probability of getting a 1 is $\frac{1}{6}$, probability of getting a 2 is $\frac{1}{6}$, probability of getting a 3 is $\frac{1}{6}$, etc. What is the sum of the probabilities of all the possible outcomes?

In general, the probability of an event happening

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

Example 3

A basket contains 5 red balls, 4 green balls and 3 blue balls. If a ball is picked at random from the basket, find:

(a) the probability of picking a blue ball.

(b) the probability of not picking a red ball.

Solution

(a) Total number of balls is 12.

The number of blue balls is 3

Therefore, P (a blue ball) $= \frac{3}{12}$

$= \frac{1}{4}$

(b) **The number** of balls which are not red is 7.

Therefore, P(not a red ball) $= \frac{7}{12}$

Example 4

Two dice are tossed together. What is the probability that the sum of the two **upper faces** will be:

(a) 7?

(b) 9?

Solution

Table 13.6

		Number on die one					
		1	2	3	4	5	6
Number on die two	1	2	3	4	5	6	⑦
	2	3	4	5	6	⑦	8
	3	4	5	6	⑦	8	⑨
	4	5	6	⑦	8	⑨	10
	5	6	⑦	8	⑨	10	11
	6	⑦	8	⑨	10	11	12

From the table there are 36 equally likely outcomes.

(a) $P(7) = \frac{6}{36} = \frac{1}{6}$

(b) $P(9) = \frac{4}{36} = \frac{1}{9}$

What is the probability that the sum of the two upper faces will be 10, 11, 4, 3, 1 ?

Example 5

A bag contains 6 black balls and some brown ones. If a ball is picked at random the probability that it is black is 0.25. Find the number of brown balls.

Solution

Let the number of balls be x.

Then the probability that a black ball is picked at random is $\frac{6}{x}$.

Therefore $\frac{6}{x} = 0.25$

$x = 24$

The total number of balls is 24.
Then the number of brown balls is 24 – 6 = 18.

Note:
The outcomes in examples 1, 2, 3, 4 and 5 can be counted.
When all possible outcomes are countable, they are said to be discrete.

Example 6
Figure 13.1 shows two concentric circles.

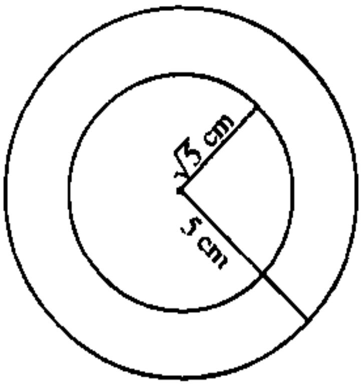


Fig 13.1

The area of the bigger circle is 25π cm^2 and that of the smaller circle is 5π cm^2. If a point is selected at random inside the bigger circle what is the probability that it lies:

(a) inside the smaller circle?
(b) outside the smaller circle?

Solution

(a) This particular point can be anywhere inside the larger circle whose area is 25π cm^2. The probability that it lies inside the smaller circle which is 5π cm^2. Therefore, the probability of the point being in the smaller circle is

$$\frac{\text{area of smaller circle}}{\text{area of larger circle}} = \frac{5\pi}{25\pi}$$

$$= \frac{1}{5}$$

(b) The area outside the smaller circle is $25\pi - 5\pi = 20\pi$ cm^2

Therefore, P (outside the smaller circle) = $\frac{20\pi}{25\pi}$

$$= \frac{4}{5}$$

Or,

$$\begin{aligned} P(\text{outside the smaller circle}) &= 1 - P(\text{inside the small circle}) \\ &= 1 - \tfrac{1}{5} \\ &= \tfrac{4}{5} \end{aligned}$$

Notice that in this example, the number of outcomes (area) cannot be listed. Such outcomes are said to be non-discrete or continuous.

Exercise 13.2

1. If a child is selected at random, what is the probability that it was born on a Monday?
2. A letter is selected at random from the English alphabet. Find the probability that it is a vowel.
3. Faces of a regular tetrahedron are marked with numbers 1, 2, 3 and 4. If the tetrahedron is tossed once, calculate the probability that the face marked 3 lands at the bottom.
4. What is the probability of getting a number that is not prime with a single toss of a die?
5. Two dice are tossed together. What is the probability that the sum of the numbers showing on their upper face is:
   (a) 3? (b) 5? (c) 8?
6. Three coins are tossed together.
   - (a) List the eight possible outcomes.
   - (b) Find the probability of getting.
     - (i) 3 heads.
     - (ii) one tail.
7. What is the probability that an integer chosen at random from the integers 1 to 20 (inclusive) is divisible by 3?
8. A bag contains 4 green balls and 5 yellow balls. If a ball is picked at random from the bag, find the probability that it is:
   - (a) a green ball
   - (b) a yellow ball
9. What is the probability of the upper face showing a number divisible by 4 when a regular dodecahedron with faces marked 1 to 12 is rolled?
10. A die is tossed and a coin span. Find the probability of getting:
    - (a) a one showing up along with a head.
    - (b) a tail showing up along with a number less than 5.
11. A card is drawn from a well shuffled ordinary pack of cards. Find the probability of drawing:
    - (a) a diamond.

(b) a red card.
(c) a queen.
(d) a 3 of hearts.

12. An urn contains green and red marbles. The probability of picking a green marble is $\frac{2}{7}$.
    (a) What is the probability of picking a red marble?
    (b) If there are 14 green marbles, what is the total number of marbles in the urn?

13. Figure 13.2 shows a spinner used in a game of chance. The arrow shown spins at random and stops at one of the eight equal sectors numbered as shown. Find the probability that the arrow points at 6.

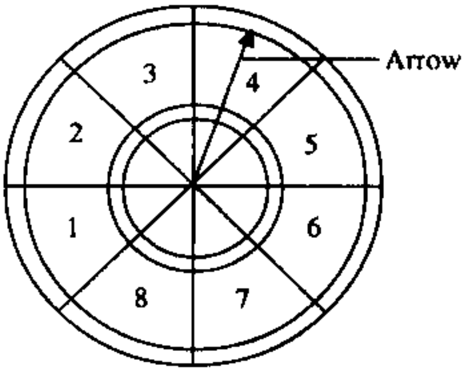


Fig. 13.2

14. The letters of the word LOITOKITOK are written on identical cards and shuffled. If a card is picked at random from the pack, find the probability that :
    (a) the card picked has a T.
    (b) the card picked has a vowel.

15. Figure 13.3 shows a circle centre O, radius r. Angle AOB is 60^0. If a point is selected at random inside the circle, find the probability that it lies in
    (a) the shaded region.
    (b) outside the shaded region.

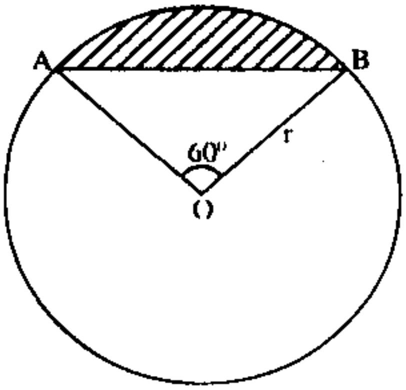


Fig. 13.3

16. Figure 13.4 shows an equilateral triangle ABC of side 4 cm. The shaded areas are sectors of circles centres at A, B and C with radii 1 cm. If a point is selected at random from the triangle, find the probability that it lies in the unshaded region.

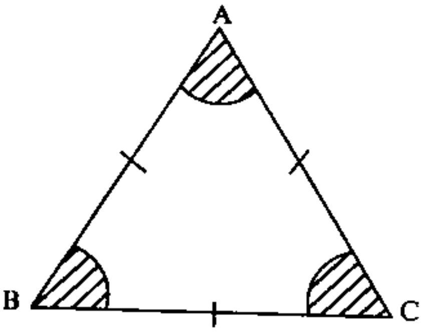


Fig. 13.4

17. A rectangle is 6 cm wide and 8 cm long. Inside this rectangle is a smaller one 3 cm wide and 4 cm long. If a point inside the larger rectangle is selected at random, find the probability that the point is:
    (a) inside the smaller rectangle.
    (b) outside the smaller rectangle.

13.6: Combined Events

So far we have dealt with simple cases where a single event can happen or fail to happen. In this section, we shall consider the probability of two or more events occurring.

Mutually Exclusive Events

When a coin is tossed once, the result will either be a head or a tail. If a head occurs, a tail cannot occur. Such events in which occurrence of one excludes the occurrence of the other are called **mutually exclusive events.**

If A and B are two mutually exclusive events, then ((A or B) = P (A) + P(B). This is the addition law of probability. For example;

(i) If a coin is tossed;

P(head or tail) = P(head) + P(tail)

$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

(ii) When a die is tossed, getting a one, a two, a three, a four, a five or a six are mutually exclusive events.

Therefore:

- $P(1 \text{ or } 2) = P(1) + P(2)$
  $= \frac{1}{6} + \frac{1}{6}$
- $P(\text{even number}) = P(2 \text{ or } 4 \text{ or } 6)$
  $= P(2) + P(4) + P(6)$
  $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
  $= \frac{1}{2}$
- $P(1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6)$
  $= P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$
  $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
  $= 1$

Example 7

A die is thrown once. Find the probability of getting:

(a) an even number or a 3. (b) an odd number or a 4.
(c) 3 or less. (d) at most 3. (e) at least 5.
(f) 4 or more. (g) not more than 2.

Solution

(a) $P(\text{an even number or a } 3) = P(\text{an even number}) + P(3)$
$= P(2 \text{ or } 4 \text{ or } 6) + P(3)$
$= P(2) + P(4) + P(6) + P(3)$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
$= \frac{2}{3}$

(b) $P(\text{an odd number or } 4) = P(\text{an odd number}) + P(4)$
$= P(1 \text{ or } 3 \text{ or } 5) + P(4)$
$= P(1) + P(3) + P(5) + P(4)$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
$= \frac{2}{3}$

(c) $P(3 \text{ or less}) = P(3 \text{ or } 2 \text{ or } 1)$
$= P(3) + P(2) + P(1)$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
$= \frac{1}{2}$

(d) P (at most 3) = P(3 or less)
= P(3 or 2 or 1)
= $\frac{1}{2}$

(e) P(at least 5) = P(5 or more)
= P(5 or 6)
= P(5) + P(6)
= $\frac{1}{6} + \frac{1}{6}$
= $\frac{1}{3}$

(f) P(4 or more) = P(at least 4)
= P(4 or 5 or 6)
= $\frac{1}{2}$

(g) P(not more than 2) = P(1 or 2)
= P(1) + P(2)
= $\frac{1}{6} + \frac{1}{6}$
= $\frac{1}{3}$

Two dice are thrown together. Find the probability of getting a sum:

(a) of 10.
(b) of 8 or 3.
(c) of at least 6.
(d) greater than 9.
(e) of at most 7.
(f) of 3 or less.
(g) not more than 3.
(h) that is even.

Independent Events

Two events A and B are said to be independent if the occurrence of A does not influence the occurrence of B, and vice versa.

Suppose that a die and a coin are thrown together. Let A be the event that the die shows a 6 and B the event that the coin shows a head. The occurrence of A does not affect the occurrence of B. Thus, A is independent of B.

If A and B are two independent events, the probability of them occurring together is the product of their individual probabilities. That is;
P(A and B) = P(A) x P(B).
This is the multiplication law of probability.

Example 8

A coin is tossed twice. What is the probability of getting a tail in both tosses?

Solution

The outcome of the 2nd toss is clearly independent of the outcome of the first.

Therefore;

$$\begin{aligned} P(T \text{ and } T) &= P(T) \times P(T) \\ &= \tfrac{1}{2} \times \tfrac{1}{2} \\ &= \tfrac{1}{4} \end{aligned}$$

Alternatively, all the possible outcomes are HH, HT, TT and TH. We are interested in the event TT.

Therefore $P(TT) = \frac{1}{4}$

Example 9

A boy throws a fair coin and a regular tetrahedron with its four faces marked 1, 2, 3 and 4. Find the probability that he gets a 3 on the tetrahedron and a head on the coin.

Solution

These are independent events.

$P(H) = \frac{1}{2}$, $P(3) = \frac{1}{4}$

Therefore;

$$\begin{aligned} P(H \text{ and } 3) &= P(H) \times P(3) \\ &= \tfrac{1}{2} \times \tfrac{1}{4} \\ &= \tfrac{1}{8} \end{aligned}$$

Alternatively, we can make a table of all the possible outcomes, as the one below.

Table 13.7

Coin

		H	T
Tetrahedron	1	1H	1T
	2	2H	2T
	3	3H	3T
	4	4H	4T

We are interested in the event 3H.

Therefore, $P(3H) = \frac{1}{8}$

Example 10

Three different machines in a factory have different probabilities of breaking down during a shift as shown in table 13.8.

Table 13.8

Machine	*Probability of breaking*
A	$\frac{4}{15}$
B	$\frac{3}{10}$
C	$\frac{2}{11}$

Find:

(a) the probability that all machines will break down during one shift.

(b) the probability that none of the machines will break down in a particular shift.

Solution

(a) P(A and B and C breaking) = P(A breaking) × P(B breaking) × P(C breaking)

$$= \frac{4}{15} \times \frac{3}{10} \times \frac{2}{11}$$

$$= \frac{24}{1650}$$

$$= \frac{4}{275}$$

(b) P(none of machines A, B and C breaks down)

= P(A and B and C do not break down)

= P(A not breaking) × P(B not breaking) × P(C not breaking)

The following have to be worked out first.

P(A does not break down) = 1 – P(A breaks down)

$$= 1 - \frac{4}{15}$$

$$= \frac{11}{15}$$

P(B does not break down) = $\frac{7}{10}$

P(C does not break down) = $\frac{9}{11}$

∴ P(A not breaking, B not breaking and C not breaking) = $\frac{11}{15} \times \frac{7}{10} \times \frac{9}{11}$

$$= \frac{21}{50}$$

Example 11

A bag contains 8 black balls and 5 white ones. If two balls are drawn from the bag, one at a time, find the probability of drawing a black ball and a white ball.

(a) without replacement.

(b) with replacement.

Solution

(a) There are only two ways we can get a black and a white ball; either drawing a white then a black, or drawing a black then a white. We need to find the two probabilities;

$$\begin{aligned} P(\text{W followed by B}) &= P(\text{W and B}) \\ &= \tfrac{8}{13} \times \tfrac{5}{12} \\ &= \tfrac{10}{39} \end{aligned}$$

$$\begin{aligned} P(\text{B followed by W}) &= P(\text{B and W}) \\ &= \tfrac{5}{13} \times \tfrac{8}{12} \\ &= \tfrac{10}{39} \end{aligned}$$

Note that the two events are mutually exclusive.

Therefore,

P(W followed by B) or (B followed by W) = P(W followed by B) + P(B followed by W)

$$\begin{aligned} &= P(\text{W and B}) + P(\text{B and W}) \\ &= \tfrac{40}{156} + \tfrac{40}{156} \\ &= \tfrac{20}{39} \end{aligned}$$

(b) Since we are replacing, the number of balls remains 13.

Therefore;

$$\begin{aligned} P(\text{W and B}) &= \tfrac{5}{13} \times \tfrac{8}{13} \\ &= \tfrac{40}{169} \end{aligned}$$

$$\begin{aligned} P(\text{B and W}) &= \tfrac{8}{13} \times \tfrac{5}{13} \\ &= \tfrac{40}{169} \end{aligned}$$

Therefore;

$$\begin{aligned} P\,[(\text{W and B}) \text{ or } (\text{B and W})] &= P\,(\text{W and B}) + P\,(\text{B and W}) \\ &= \tfrac{40}{169} + \tfrac{40}{169} \\ &= \tfrac{80}{169} \end{aligned}$$

Example 12

Two marbles are drawn in turn from a pack containing 3 red marbles, 6 white **marbles,** 7 black marbles and 9 green marbles.

(a) If this is done with replacement, determine the probability of drawing:

- (i) two white marbles.
- (ii) a black then a green ball.
- (iii) no red marble.

(b) Repeat the question for no replacement.

Solution

(a) (i) Probability of drawing a white marble = $\frac{6}{25}$

Since there is replacement, the white marble is replaced and probability of drawing a white marble on the second drawn is $\frac{6}{25}$.

$\therefore$ the probability of drawing a white marble on the first draw and a white marble on the second draw is; $\frac{6}{25} \times \frac{6}{25} = \frac{36}{625}$

(ii) The probability of drawing a black marble is $\frac{7}{25}$ and the probability of drawing a green marble is $\frac{9}{25}$.

$\therefore$ the probability of drawing a black and a green marble is $\frac{7}{25} \times \frac{9}{25} = \frac{63}{625}$

(iii) The probability of drawing a non-red marble is $\frac{22}{25}$

$\therefore$ the probability of drawing a white, black or green marble on both the first and second draws with replacement is $\frac{22}{25} \times \frac{22}{25} = \frac{484}{625}$

Alternatively,

P(non-red) = P(white or black or green) and P(white or black or green)

$$= \frac{22}{25} \times \frac{22}{25}$$

$$= \frac{484}{625}$$

(b) If the drawing of the marbles is done without replacements:

(i) $\frac{6}{25} \times \frac{5}{24} = \frac{1}{20}$

(ii) $\frac{7}{25} \times \frac{9}{24} = \frac{21}{200}$

(iii) $\frac{22}{25} \times \frac{21}{24} = \frac{77}{100}$

Example 13

Kamau, Njoroge and Kariuki are practising archery. The probabilty of Kamau hitting the target is $\frac{2}{5}$, that of Njoroge hitting the target is $\frac{1}{4}$ and that of Kariuki hitting the target is $\frac{3}{7}$. Find the probability that in one attempt:

(a) only one hits the target.
(b) all three hit the target.
(c) none of them hits the target.
(d) two hit the target.
(e) at least one hits the target.

Solution

(a) P(only one hits the target)

= P(one of the men hit and the other two miss)

P(only Kamau hits and other two miss) $= \frac{2}{5} \times \frac{3}{4} \times \frac{4}{7}$

$= \frac{6}{35}$

P(only Njoroge hits and other two miss) $= \frac{1}{4} \times \frac{3}{5} \times \frac{4}{7}$

$= \frac{3}{35}$

P(only Kariuki hits and other two miss) $= \frac{3}{7} \times \frac{3}{5} \times \frac{3}{4}$

$= \frac{27}{140}$

P(only one hits) = P(Kamau hits or Njoroge hits or Kariuki hits)

$= \frac{6}{35} + \frac{3}{35} + \frac{27}{140}$

$= \frac{9}{20}$

(b) P(all three hit) $= \frac{2}{5} \times \frac{1}{4} \times \frac{3}{7}$

$= \frac{3}{70}$

(c) P(none hits) $= \frac{3}{5} \times \frac{3}{4} \times \frac{4}{7}$

$= \frac{9}{35}$

(d) P(two hit the target) is the probability of:

Kamau and Njoroge hit the target and Kariuki misses $= \frac{2}{5} \times \frac{1}{4} \times \frac{4}{7}$

Njoroge and Kariuki hit the target and Kamau misses $= \frac{1}{4} \times \frac{3}{7} \times \frac{3}{5}$

or

Kamau and Kariuki hit the target and Njoroge misses = $\frac{2}{5} \times \frac{3}{7} \times \frac{3}{4}$

$\therefore$ P(two hit the target) $= (\frac{2}{5} \times \frac{1}{4} \times \frac{4}{7}) + (\frac{1}{4} \times \frac{3}{7} \times \frac{3}{5}) + (\frac{2}{5} \times \frac{3}{7} \times \frac{3}{4})$

$= \frac{8}{140} + \frac{9}{140} + \frac{18}{140}$

$= \frac{1}{4}$

(e) P(at least one hits the target) = P(1 or 2 or 3 of the players hit the target)

$= \frac{9}{20} + \frac{1}{4} + \frac{3}{70}$

$= \frac{63}{140} + \frac{35}{140} + \frac{6}{140}$

$= \frac{26}{35}$

Alternatively;

P(at least one hits the target = 1 – P(none hits the target)

$= 1 - \frac{9}{35}$

$= \frac{26}{35}$

Note:

P(one hits the target) is different from P(at least one hits the target).

Exercise 13.3

1. Three fair coins are tossed together. Find the probability of getting one head and two tails.
2. The probability of Kilonzo hitting a target with an arrow is $\frac{2}{3}$. What is the probability of him missing the target in four consecutive attempts?
3. The probabilities of Juma, Ali and Musa passing an examination are $\frac{1}{3}$, $\frac{3}{5}$ and $\frac{1}{4}$ respectively. Find the probability that:
   (a) either Juma or Ali will pass.
   (b) both Ali and Musa will pass.
   (c) all the three will fail.
4. Two balls are drawn in turn without replacement from a box containing three green and two white balls. Find the probability of drawing:
   (a) two green balls.
   (b) two white balls.
   (c) two balls of the same colour.
   (d) two balls of different colours.

(e) at least one green ball.

5. An integer is picked at random from the integers 1 to 20 (inclusive). Find the probability that:
   (a) It is divisible by 4.
   (b) It is not divisible by 4.
   (c) It is divisible by 2 or 4 or both.
   (d) It is divisible by 2 and 3.
6. If a certain unfair coin is tossed, the probability of obtaining a tail is 25%. Find:
   (a) the probability of obtaining a head when the coin is tossed once.
   (b) the probability of obtaining at least one tail when the coin is tossed twice.
7. Six cards are numbered 1, 1, 2, 3, 3 and 4. Two cards are picked at random, one after the other without replacement. Find the probability that the two cards picked are:
   (a) both are numbered 1.
   (b) such that one is numbered 1 and the other is numbered 3.
8. A committee of three people is to be chosen at random from three men and two women. Find the probability that:
   (a) all the three people chosen are men.
   (b) one of the three people chosen is a woman.
9. A bag contains four black (B) and three red (R) balls. The balls are drawn from the bag without replacement. Find the probability of drawing the balls in the order B, R, B, R, B, R and B.
10. In a doctor's waiting room there are four men and three women. Find the probability that:
    (a) the first three people called in to see the doctor will be a woman, man and woman in that order, and,
    (b) the first three people called in to see the doctor will be men, if all patients have equal chances of seeing the doctor.
11. Out of 30 bolts in a box, seven are defective. If a mechanic selects three bolts at random from the box, find the probability that none of them will be defective.
12. A fair die is thrown two times. Find the probability of obtaining:
    (a) the same number both times.
    (b) two different numbers.
    (c) one odd number and one even number.
13. Two players X and Y play a game. The probability of X and Y winning the game is 0.4 and 0.25 respectively. If neither X nor Y wins, the game is drawn, find the probability that the game is:

  - (a) drawn.
  - (b) either won by Y or drawn.

14. The probability that Jane wins a game is 0.6. She plays the game until she loses. Determine the probability that she will not play the fifth game.

13.7: Tree diagrams

A tree diagram is a way of enumerating the probability space.

Suppose that a coin is tossed twice. The tree diagram (figure 13.5) below shows probability space.

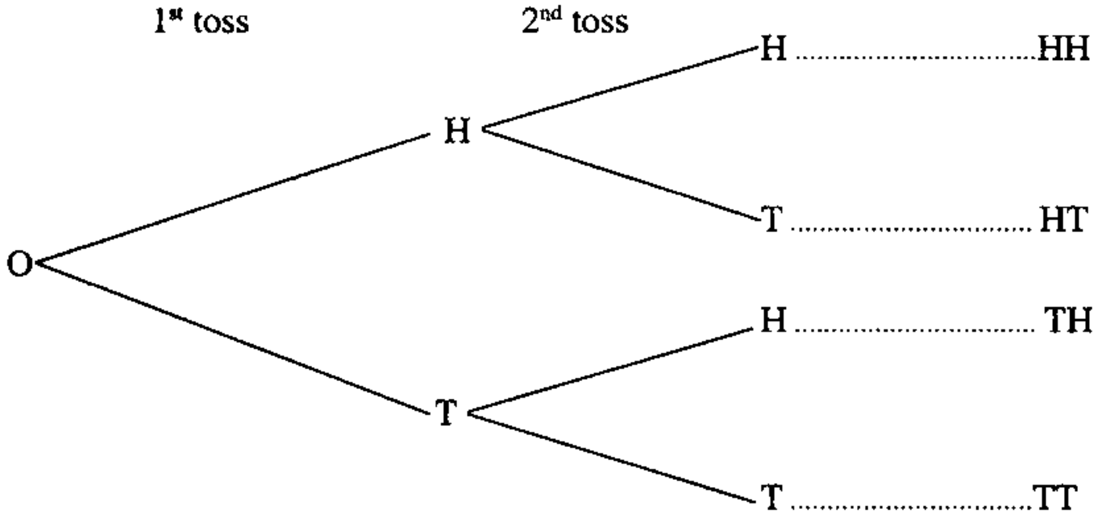


Fig. 13.5

By following the branches of the tree from the starting point O, we can pick out four equally likely outcomes. These are HH, HT, TH and TT.

The probabilities of mutually exclusive events and those of independent events are best done by the use of a tree diagram as illustrated by the following example.

Example 14

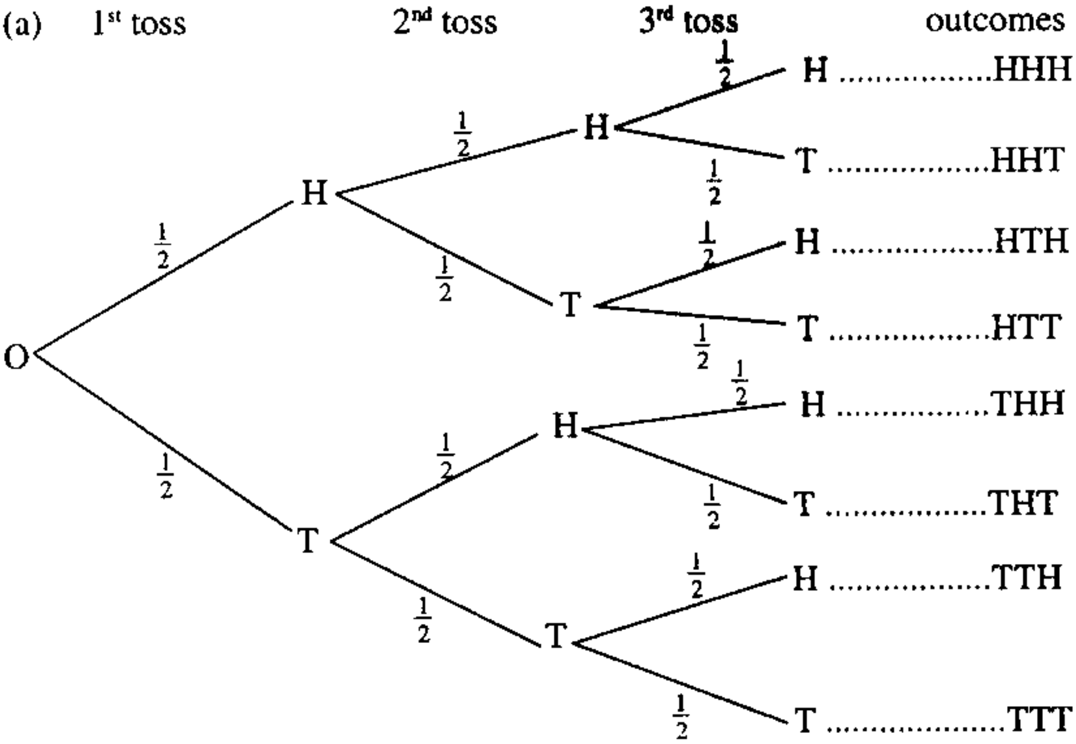
A coin is tossed three times.

(a) Draw a tree diagram to show all the possible outcomes.
(b) Find the probability of getting:
  - (i) one head.
  - (ii) two heads and a tail, in that order.
  - (iii) two heads and a tail, in any order.
  - (iv) three heads.
  - (v) at least one head.
  - (vi) no head.

Solution

Figure 13.6 shows all the 8 equally likely possible outcomes. The probability of each event is written on each branch of the tree. These possibilities are HHH, HHT, HTH, HTT, THH, THT, TTH and TTT.

(a)



(b) (i) P (one head) $= P(HTT + P(THT + P(TTH)$

$$= (\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}) + (\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}) + (\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2})$$

$$= \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}$$

$$= \tfrac{3}{8}$$

(ii) 1 P(two heads and a tail in that order) $= P(HHT)$

$$= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}$$

$$= \tfrac{1}{8}$$

(iii) P(two heads and a tail in any order) $= P(HHT) + P(HTH) + P(THH)$

$$= \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}$$

$$= \tfrac{3}{8}$$

(iv) P (three heads) $= P(HHH)$

$$= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}$$

$$= \tfrac{1}{8}$$

(v) P(at least one head) = P(1 or 2 or 3 heads)

$$= P(1) + P(2) + P(3)$$

$$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= \frac{7}{8}$$

(vi) P(no head) = P(TTT)

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

Or, P(no head) = 1 – P(a least one head occurs)

$$= 1 - \frac{7}{8}$$

$$= \frac{1}{8}$$

Example 15

The probability that Omweri goes to Nakuru is $\frac{1}{4}$. If he goes to Nakuru, the probability that he will see a flamingo is $\frac{1}{2}$. If he does not go to Nakuru, the probability that he will see a flamingo is $\frac{1}{3}$. Find the probability that:

(a) Omweri will go to Nakuru and see a flamingo.
(b) Omweri will not go to Nakuru yet he will see a flamingo.
(c) Omweri will see a flamingo.

Solution

Let N stand for going to Nakuru, N′ stand for not going to Nakuru, stand for seeing a flamingo and F′ stand for not seeing a flamingo.

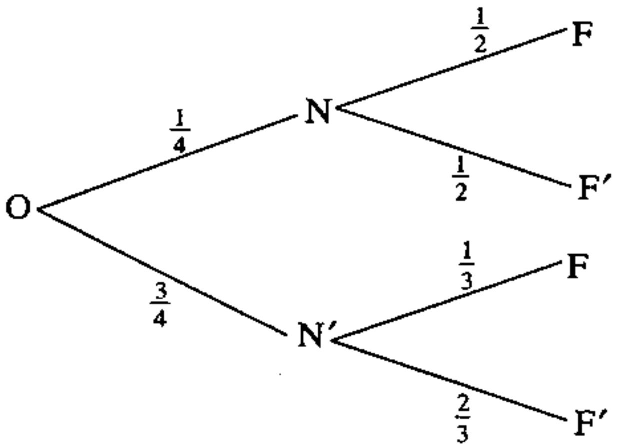


Fig. 13.7

(a) P(He goes to Nakuru and sees a flamingo) $= P(N \text{ and } F)$

$$= P(N) \times P(F)$$

$$= \tfrac{1}{4} \times \tfrac{1}{2}$$

$$= \tfrac{1}{8}$$

(b) P(He does not go to Nakuru and yet sees a flamingo) $= P(N') \text{ x } P(F)$

$$= P(N' \text{ and } F)$$

$$= \tfrac{3}{4} \times \tfrac{1}{3}$$

$$= \tfrac{1}{4}$$

(c) P(He sees a flamingo) $= P(N \text{ and } F) \text{ or } P(N' \text{ and } F)$

$$= P(N \text{ and } F) + P(N' \text{ and } F)$$

$$= \tfrac{1}{8} + \tfrac{1}{4}$$

$$= \tfrac{3}{8}$$

Exercise 13.4

Use tree diagrams to solve the problems in this exercise.

1. A fair coin is tossed 4 times. Find the probability of getting:
   (a) 4 tails.
   (b) 3 heads and 1 tail.
   (c) at least 2 heads.
2. The probability that horse A wins a race against B is $\frac{2}{5}$. If the horses run 3 races in succession, find the probability that:
   (a) horse B wins all the races.
   (b) horse A wins only 1 race.
   (c) horse A wins at least 1 race.
   (d) horse B wins more races than horse A.
3. If two cards are chosen from an ordinary pack with replacement, what is the probability of getting at least one heart?
4. A boy tosses a coin twice, and if the last result is a tail, he throws a die
   (a) How many possible final outcomes are there?
   (b) What is the probability of getting a head, tail and a 6 in that order.
5. In a game of snakes and ladders played with one die, a player has to throw a 6 in order to start. What is the probability that a player will start after two throws at most?
6. A man playing a game of darts has a probability of $\frac{1}{3}$ of hitting a double. If he throws 3 darts, find the probability of him getting:

(a) 2 doubles.
(b) more than one double.
(c) at least 2 doubles.

7. Ten per cent of transistors manufactured by a company are defective. If five transistors are chosen at random, find the probability that:
   (a) 1 will be defective.
   (b) 2 will be defective.
   (c) 3 or more will be defective.
8. Five fair coins are tossed at the same time. Find the probability of getting:
   (a) at most 4 tails.
   (b) at least 5 heads.
   (c) 3 heads and 2 tails.
9. The probability that a day is rainy is $\frac{1}{4}$. The probability that I carry an umbrella on a rainy day is $\frac{1}{7}$ and that I carry an umbrella on a non-rainy day is $\frac{2}{7}$. Find the probability that:
   (a) It will not be rainy and 1 carry an umbrella.
   (b) I shall carry an umbrella.
10. In a certain town, 15% of the people are bald-headed. If 4 people are chosen at random, find the probability that:
   (a) 2 are bald-headed.
   (b) less than 3 are bald-headed.
   (c) 3 are not bald-headed.
   (d) at least 2 are bald-headed.
11. Two teams Kifaru and Simba are playing in a tournament. The winner is the first team to win either two consecutive games or a total of three games. Copy and complete the tree diagram below to show the possible ways in which the tournament may end.

Let S stand for Simba
Let K stand for Kifaru

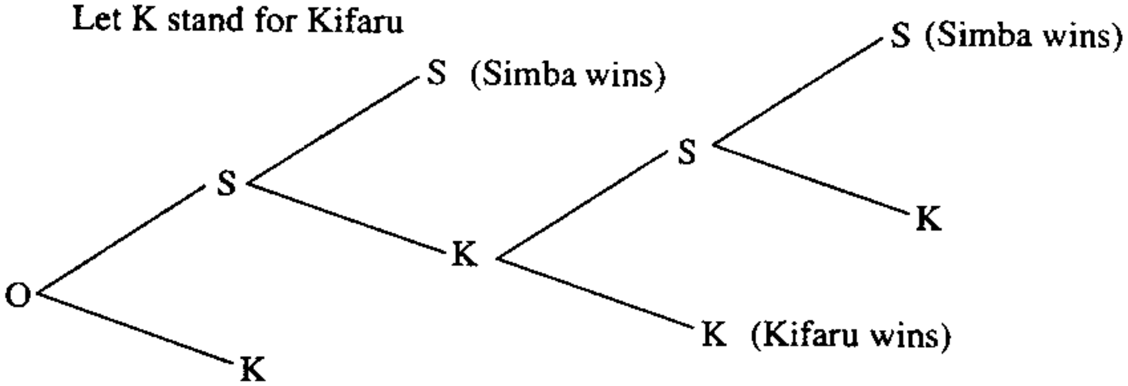


Fig. 13.8

If in any one game the probability that either team wins is $\frac{1}{2}$, find the probability that:

(a) the tournament ends after only two games.

(b) five games need to be played before the winner is decided.

(c) Simba wins the tournament.

12. A bag contains black, white and red beads of the same type in the ratio 6 : 4 : 3 respectively. A bead is picked at random without replacement and its colour noted.

    (a) Find the probability that the first bead picked is:

        (i) black.

        (ii) white or red.

    (b) Using a tree diagram, determine the probability that:

        (i) the first two beads picked are both white.

        (ii) only one of the first two beads picked is red.

13. A, B and C are 3 bags. A contains 2 blue beads and 3 red ones, B contains 5 blue beads and 4 red ones and C contains 3 white beads. A bag is chosen at random and a bead drawn from it. Find the probability that:

    (a) it is red.

    (b) it is blue.

    (c) it is white.

14. In Hekima College, there are 120 female students and 80 male students. The probability of a male student completing a course is 0.75 and that of a female not completing the same course is 0.35. A student is chosen at random. Find the probability that:

    (a) he is male and completes the course.

    (b) the student will not complete the course.

Chapter Fourteen

COMPOUND PROPORTIONS AND RATES OF WORK

14.1: Compound Proportions

Consider the numbers 3, 6, 17 and 34. We notice that 3 : 6 = 1 : 2 and 17 : 34 = 1 : 2. Thus, the two ratios are equal, i.e., 3 : 6 = 17 : 34 . The numbers 3, 6, 17 and 34 are therefore said to be in proportion.

In general, any four numbers a, b, c and d are in proportion if;

$\frac{a}{b} = \frac{c}{d}$. Thus, if a, b, c and d are in proportion;

$\frac{a}{c} = \frac{b}{d}$

Example 1

Find the value of a that makes 4, 10, a and 50 to be in proportion.

Solution

Since 4, 10, a and 50 are in proportion;

$\frac{4}{10} = \frac{a}{50}$

$\therefore\ 10a = 4 \times 50$

$a = \frac{4 \times 50}{10}$

$= 20$

Continued Proportions

Consider the numbers 2, 4, 8 and 16. Taking the ratios of successive terms, we have;

$\frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \frac{1}{2}$

The four numbers are said to be in continued proportion.

In general, if a, b, c and d are in continued proportion;

$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

Thus, if the numbers are continued proportion, they form geometric progression. (since they have a constant ratio).

For example, 3, 6, 12 and 24 form a geometric progression and they are in continued proportion.

Identify which of the sequences below are in continued proportion:

(i) 8, 12, 18, 27
(ii) 5, 7, 9, 11
(iii) 32, −16, 8, −4
(iv) 41, 42, 43, 44

If a, b and c are in continued proportion, then,

$$\frac{a}{b} = \frac{b}{c}$$

$$\therefore b^2 = ac$$

$$b = \pm\sqrt{ac}$$

b is called the mean proportional to a and c. c is called the third proportional to a and b.

Example 2

Find the mean proportional to 3 and 6.

Solution

Let the mean proportional to 3 and 6 be a. Then, 3, a and 6 are in continued proportion.

$$\therefore \frac{3}{a} = \frac{a}{6}$$

$$a^2 = 18$$

$$a = \pm\sqrt{18}$$

$$= \pm\sqrt{2 \times 9}$$

$$= \pm 3\sqrt{2}$$

Since 3 and 6 are positive, then the mean proportional is $3\sqrt{2}$.

If a, b, c and d are in continued proportion, then a and d are extremes and b and c are the means. Now;

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$$

$\rightarrow a = bk$, $b = ck$ and $c = dk$

Also $\frac{a}{b} = \frac{b}{d}$

Example 3

Given that x : y = 2 : 3, find the ratio (5x − 4y) : (x + y).

Solution

Since x : y = 2 : 3;

$\frac{x}{2} = \frac{y}{3} = k, \Rightarrow x = 2k$ and $y = 3k$.

$$
\begin{aligned}
(5x - 4y) : (x + y) &= (10k - 12k) : (2k + 3k) \\
&= -2k : 5k \\
&= -2 : 5
\end{aligned}
$$

Alternatively;

$x : y = 2 : 3 \Rightarrow \frac{x}{y} = \frac{2}{3}$

$x = \frac{2}{3}y$

$$
\begin{aligned}
5x - 4y &= \tfrac{10}{3}y - 4y \\
&= -\tfrac{2}{3}y
\end{aligned}
$$

$$
\begin{aligned}
x + y &= \tfrac{2}{3}y + y \\
&= \tfrac{5}{3}y
\end{aligned}
$$

$$
\begin{aligned}
\therefore 5x - 4y : x + y &= -\tfrac{2}{3}y : \tfrac{5}{3}y \\
&= -2 : 5
\end{aligned}
$$

Example 4

If $\frac{a}{b} = \frac{c}{d}$, show that $\dfrac{a - 3b}{b - 3a} = \dfrac{c - 3d}{d - 3c}$.

Solution

$\frac{a}{b} = \frac{c}{d} \implies \frac{a}{c} = \frac{b}{d}$.

$\frac{a}{c} = \frac{b}{d} = k, \implies a = kc, b = kd$

Substituting kc for a and kd for b in the expression $\dfrac{a - 3b}{b - 3a}$;

$$
\begin{aligned}
\frac{kc - 3kd}{kd - 3kd} &= \frac{k(c - 3d)}{k(d - 3c)} \\
&= \frac{c - 3d}{d - 3c}
\end{aligned}
$$

$$\therefore \frac{a - 3b}{b - 3a} = \frac{c - 3d}{d - 3c}$$

Alternatively;

$\frac{a}{b} = \frac{c}{d} \implies a = \frac{bc}{d}$

$$\therefore \frac{a - 3b}{b - 3a} = \frac{\frac{bc}{d} - 3b}{b - \frac{3bc}{d}}$$

$$= \frac{\frac{bc - 3bd}{d}}{\frac{bd - 3bc}{d}}$$

$$= \frac{bc - 3bd}{bd - 3bc}$$

$$= \frac{b(c - 3d)}{b(d - 3c)}$$

$$= \frac{c - 3d}{d - 3c}$$

Exercise 14.1

1. Find the fourth number if the following are in proportion:
   (a) 3, 5, 6 (b) 6, 11, 14 (c) 1.5, 2, 2.5
   (d) 25, 64, 81 (e) p^2, pq, pr
2. Find the third proportional to:
   (a) $\sqrt{5}$, $\sqrt{10}$ (b) $\sqrt{10}$, $\sqrt{5}$ (c) 36, 48
   (d) 4, 3 (e) x, y
3. Find the mean proportional between:
   (a) 6 and 24 (b) 9 and 1 (c) 2 and 5
   (d) a^2b and b^3 (e) $\frac{n}{m}$ and $\frac{m}{n}$
4. If $\frac{c - d}{c + d} = \frac{5}{11}$, find he ratio c : d.
5. If p, q, r and s are in continued proportion, show that:
   (a) $\frac{2p + r}{2q + s} = \frac{p - 3r}{q - 3s}$ (b) $\frac{p^2 + r^2}{q^2 + s^2} = \frac{pr}{qs}$
6. If x : y = 5 : 3, find the value of (x + y) : (2y – x)
7. If $\frac{2x + y}{3x + 4y} = \frac{2}{5}$, find the value of $\frac{x}{y}$.
8. If $\frac{4a - b}{a + b} = 3$, find the ratio of a : b and the value of $\frac{2a}{3b}$.

14.2: Proportional Parts

Consider a school with a population of 1 200, the ratio of boys and girls being 2 : 3. In this case, the population is divided into (2 + 3), i.e., 5 proportional parts. Two out of the five represent boys while 3 out of the five represent girls.

Therefore, number of boys $= \frac{2}{5} \times 1\,200$

$= 480$

and number of girls $= \frac{3}{5} \times 480$

$= 720$

Suppose a total amount of sh. 1 000 is to be divided in the ratio 1 : 2 : 3 : 4, then the parts of 1 000 proportional to 1, 2, 3 and 4 are;

$$\frac{1}{1+2+3+4} \times 1\,000 = 100$$

$$\frac{2}{1+2+3+4} \times 1\,000 = 200$$

$$\frac{3}{1+2+3+4} \times 1\,000 = 300$$

$$\frac{4}{1+2+3+4} \times 1\,000 = 400$$

In general, if n is to be divided in the ratio a : b : c, then the parts of n proportional to a, b and c are $\frac{a}{a+b+c} \times n$, $\frac{b}{a+b+c} \times n$ and $\frac{c}{a+b+c} \times n$ respectively.

Example 5

Onyango, Juma and Cheptoo shared sh. 270 000 in the ratio 2 : 3 : 4 respectively. How much did each get?

Solution

The parts of sh. 270,000 proportional to 2, 3 and 4 are

$\frac{2}{9} \times 270\,000$, $\frac{3}{9} \times 270\,000$, $\frac{4}{9} \times 270\,000$

$\frac{2}{9} \times 270\,000 =$ sh. 60 000

$\frac{3}{9} \times 270\,000 =$ sh. 90 000

$\frac{4}{9} \times 270\,000 =$ sh. 120 000

$\therefore$ Onyango got sh. 60 000
Juma got sh. 90 000
Cheptoo got sh. 120 000

Alternatively, using unitary method,
9 parts represent sh. 270 000

1 part represents sh. $\frac{270\ 000}{9}$

$\therefore$ 2 parts represent sh. $\frac{270\ 000}{9} \times 2 =$ sh. 60 000

3 parts represent sh. $\frac{270\ 000}{9} \times 3 =$ sh. 90 000

4 parts represent sh. $\frac{270\ 000}{9} \times 4 =$ sh. 120 000

Exercise 14.2

1. Divide
   (a) sh. 1 050 into 2 parts in the ratio 5 : 9.
   (b) 306 kg into 3 parts in the ratio 1 : 3 : 5
   (c) 880 ha into 4 parts in the ratio 4 : 5 : 6 : 7
   (d) 539 litres into 3 parts in the ratio 16 : 6 : 27
2. Divide 360^0 into 3 angles in the ratio 3 : 5 : 7.
3. A triangular ranch has a perimeter of 157.5 Km. Find the length of each side if the sides are in the ratio 4 : 7 : 10.
4. Four metres of a fishing co-operative society hold 450, 720, 1 020 and 1 350 shares. They decided to divide their profits of sh. 594 720 at the end of the year in proportion to their shares. Find the amount each got.
5. An alloy is made up of copper, tin and phosphorus in the ratio 5 : 14 : 1. Find the mass of copper in 185 kg of the alloy.
6. Two farmers, Limo and Koech, share the grazing of a field. Limo puts in 6 cows for 10 days and Koech puts in 15 cows for 16 days. The owner of the farm charges a rent of sh. 10 800. Find the amount of money each pays to the owner if Limo and Koech share the rent proportionally.
7. A man begins to trade with sh. 37 100. Later on, a partner joins him, injecting sh. 63 600 into business. At the end of the first year's trading, the two partners share the profits equally. Find the time of the year when the second partner joined the first partner.
8. A man takes a five-litre tin of white paint, a three-litre tin of yellow paint and a two-litre tin of blue paint and mixes the contents together in a bucket. If he now fills the original three tins with the mixture, how much white paint will there be in the two-litre tin?
9. Three partners Otieno, Ouma and Makotsi decide to start an export business together. If Otieno contributes sh. 20 000 for 9 months, Ouma contributes sh. 25 000 for 6 months and Makotsi contributes sh. 35 000 for 4 months. Find their share in profits of sh. 1 310 000.

14.3: Rates of Work and Mixtures

Example 6

20 men can lay 36 m of pipe in 8 hours. How long would 25 men take to lay the next 54 m of pipe?

Solution

20 men can lay 36 m in 8 days.

1 man can lay 36 m in 8×20 h.

1 man can lay 1 m in $\frac{8 \times 20}{36}$ h.

25 men can lay 1 m in $\frac{8 \times 20}{36 \times 25}$ h.

$$\text{25 men to lay 54 m require } = \frac{8 \times 20 \times 54}{36 \times 25} \text{ h}$$

$$= \frac{48}{5} \text{ h}$$

$$= 9\tfrac{3}{5}\text{h}$$

Alternatively, consider, 20 men each working 8 hours as 20×8 man-hours.

∴ 36 m requires 20×8 man-hours

1 m requires $\frac{20 \times 8}{36}$ man-hours

54 m requires $\frac{20 \times 8 \times 54}{36}$ man-hours

$$\text{Since there are 25 men, number of hours } = \frac{20 \times 8 \times 54}{36 \times 25} \text{ h}$$

$$= 9\tfrac{3}{5} \text{ h}$$

Example 7

Three people A, B and C can do a piece of work in 45 hours, 40 hours and 30 hours respectively. How long can B take to complete the work when he starts after A and C have worked for 13 hours each?

Solution

A can do $\frac{1}{45}$ of the work in 1 h.

B can do $\frac{1}{40}$ of the work in 1 h.

C can do $\frac{1}{30}$ of the work in 1 h.

In 1 hour, A and C will do ($\frac{1}{45} + \frac{1}{30} = \frac{1}{18}$) of work

For 13 hours, A and C will have done $\frac{1}{18} \times 13 = \frac{13}{18}$ of the work.

Amount of work remaining is $1 - \frac{13}{18} = \frac{5}{18}$ of the work.

B can do the remaining piece of work in $\left(\frac{5}{18} \div \frac{1}{40}\right)$h $= 11\frac{1}{9}$ h

Example 8

Tap P can fill a tank in 2 hrs and tap Q can fill the same tank in 4 hrs. Tap R can empty the tank in 3 hrs.

(a) If tap R is closed, how long would it take taps P and Q to fill the tank?

(b) Calculate how long it would take to fill the tank when the three taps P, Q and R are left running?

Solution

(a) Tap P fills $\frac{1}{2}$ of the tank in 1 h.

Tap Q fills $\frac{1}{4}$ of the tank in 1 h.

Tap R empties $\frac{1}{3}$ of the tank in 1 h.

In 1 h, P and Q fill $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of tank.

$\therefore$ $\frac{3}{4}$ of tank is filled in 1 h.

Time taken to fill the tank $\left(\frac{4}{4}\right) = \left(\frac{4}{4} \div \frac{3}{4}\right)$h

$= \frac{4}{3}$h

(b) In 1 h, P and Q fill $\frac{3}{4}$ of tank while R empties $\frac{1}{3}$ of the tank.

When all taps are open, $\left(\frac{1}{2} + \frac{1}{4} - \frac{1}{3} = \frac{5}{12}\right)$ of tank is filled in 1h.

$\frac{5}{12}$ of tank is filled in 1h.

$\therefore$ time required to fill the tank $\left(\frac{12}{12}\right) = \left(\frac{12}{12} \div \frac{5}{12}\right) \times 1$h

$= 2\frac{2}{5}$h

Example 9

In what proportion should grades of sugar costing sh. 45 and sh. 50 per kilogram be mixed in order to produce a blend worth sh. 48 per kilogram?

Solution

Let n kilograms of the grade costing sh. 45 per kg be mixed with 1 kilogram of grade costing sh. 50 per kg.

Total cost of the two blends is sh. (45n + 50).

The mass of the mixture is (n + 1) kg.

∴ Total cost of the mixture is sh. (n + 1)48.

$$45n + 50 = 48(n + 1)$$

$$45n + 50 = 48n + 48$$

$$50 = 3n + 48$$

$$2 = 3n$$

$$n = \frac{2}{3}$$

The two grades are mixed in the proportion $\frac{2}{3} : 1 = 2 : 3$

Alternatively;

Let x kg of grade costing sh. 45 per kg be mixed with y kg of grade costing sh. 50 per kg. Total cost will be sh. (45x + 50y).

∴ Cost per kg of mixture is sh. $\frac{45x + 50y}{x + y}$

$$\therefore \frac{45x + 50y}{x + y} = 48$$

$$45x + 50y = 48(x + y)$$

$$45x + 50y = 48x + 48y$$

$$2y = 3x$$

$$\frac{x}{y} = \frac{2}{3}$$

The proportion is $x : y = 2 : 3$

Exercise 14.3

1. Three types of tea leaves costing sh. 26, sh. 28 and sh. 32 per packet are mixed in the ratio 1 : 3 : 5 respectively. Find the cost of the mixture per packet.
2. Tractor A takes 4 days to mow a field while tractor B takes 7 days to mow the same field. How long will it take both tractors to do the work ?
3. Find the cost of 6 kg of a mixture of maize flour and millet flour, if 5 kg of maize flour costing sh. 75.00 is mixed with 3 kg of millet flour costing sh. 120.
4. A bag contains 27 dozen tennis balls. Some are bought at sh. 173.20 a dozen and the rest at sh. 184 a dozen. The average price is sh. 15 per ball. How many balls are there for each kind?
5. Two men each working for 8 hours a day can cultivate an acre of land in 4 days. How long would 6 men, each working 4 hours a day, take to cultivate 4 acres?

6. A sum of money is enough to pay a man's wages for two weeks and the same amount is enough to pay another man for three weeks. For how long will it take to pay the wages of the two men together?
7. Three people can use a tin of flour in seven days. How long will it take 5 people to complete the tin of flour if they eat at the same rate?
8. Two types of juices contain 85% and 60% of water. In what ratio must the two be mixed so that the mixture contains 75% water?
9. 2 000 litres of crude oil contains three ingredients, A, B and C. The proportion by volume of A to B is 3 : 4 and B: C is 1 : 2. Which of the ingredients is of the highest proportion in the mixture?
10. A mixture of sand, cement and ballast is in the ratio of 5 : 1 : 3. If the cost of 7 tonnes of sand is sh. 3 000, 5 tonnes of cement sh. 50 000 and 8 tonnes of ballast sh. 4000, find the cost of 7 tonnes of the mixture.
11. If a cold water tap fills a bath tub in 6 minutes and a hot water tap fills it in 8 minutes, how long would they take to fill the tub if both were turned on simultaneously?
12. Paint A costs sh. 150 per litre while B costs sh. 160 per litre. In what proportion must A be mixed with B to produce a mixture costing sh. 156 per litre?
13. Tap P fills a cistern in 5 minutes and tap Q empties it in 3 minutes. Starting with a full cistern, how long will it take to empty it if both taps are open at the same time?
14. If 5 men can erect 2 cottages in 21 days, how many more men, working at the same rate, will be needed to construct 6 cottages in the same period?
15. Three taps A, B and C can fill a water tank in 30 min, 25 min and 15 min respectively. If the three taps are turned on for 5 min, then A and C are closed, how long would it take before the tank is full?
16. In order to make a certain alloy, 9 kg of metal A and 16 kg of metal B are melted to form the mixture. If the densities of A and B are 800 kg/m^3 and 7 860 kg/m^3 respectively, calculate the ratio of A to B by volume in the alloy in the form 1 : n.
17. Two types of oranges are bought, one at 5 for sh. 4 and the other at 4 for sh. 10. The oranges are then mixed in such a way that there are 3 of the first type to every 2 of the second type and the mixture is sold for sh. 7. Find the percentage loss or gain.
18. Mueni, Akeyo and Cheruto working together take 30 minutes to make *chapatis*. Mueni and Cheruto together take 40 minutes while Mueni and Akeyo together take 45 minutes. How long would each one of them take to make the same number of *chapatis*?

19. The total expenditure in a hotel for 12 men for 15 days is sh. 72 000. The expenditure in the same hotel for 8 men for 10 days is sh. 32 000. Find the average cost per man per day for all together.
20. When draining a swimming pool containing 56 556 litres of water, a pipe of uniform cross-section are of 78.55 cm^2 is used. It takes 10 hours to drain the pool. If the pipe is full, calculate the rate at which the water is flowing through it.
21. Miriti and Mwamburi working together can do a piece of work in $2\frac{2}{5}$ days. Miriti working alone takes 2 days less than Mwamburi. How long does it take Mwamburi to do the work alone?
22. A tea blender buys two grades of tea at sh. 60 and sh. 80 per packet. Find the ratio in which he should mix them so that by selling the mixture at sh. 90.00, a profit of 25% is realised.

Chapter Fifteen

GRAPHICAL METHODS

15.1: Tables and Graphs of Given Relations

Graphs can be used to represent various mathematical functions. Consider the following.

Example 1

Draw the graph of $y = 3\sqrt{x}$ for $0 \leq x \leq 9$.

Solution

Table 15.1

x	0	1	2	3	4	5	6	7	8	9
$\sqrt{x}$	0	1	1.41	1.73	2	2.24	2.45	2.65	2.83	3
$y = 3\sqrt{x}$	0	3	4.23	5.19	6	6.72	7.35	7.95	8.49	9

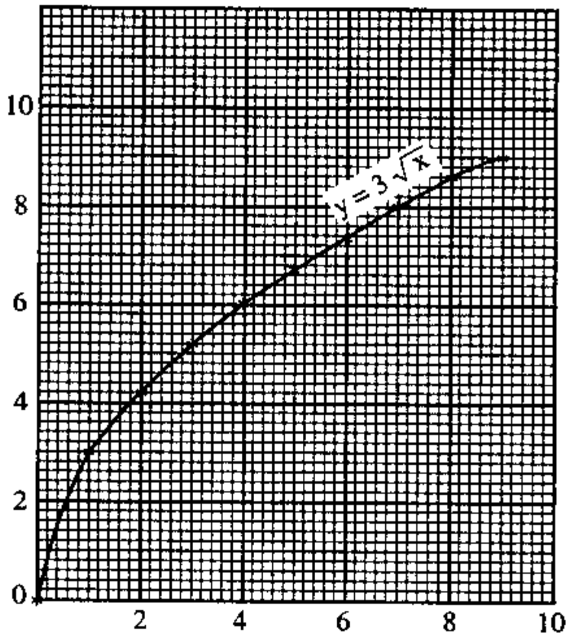


Fig. 15.1

Example 2

Draw the graph of $y = \frac{2}{x} + 1$. Use the graph to find the value of y when $x = 1.6$.

Solution

x	–4	–3	–2	–1	0	1	2	3	4
$\frac{2}{x}$	–0.5	–0.67	–1	–2	∞	2	1	0.67	0.5
$y = \frac{2}{x}+1$	0.5	0.33	0	–1	∞	3	2	1.67	1.5

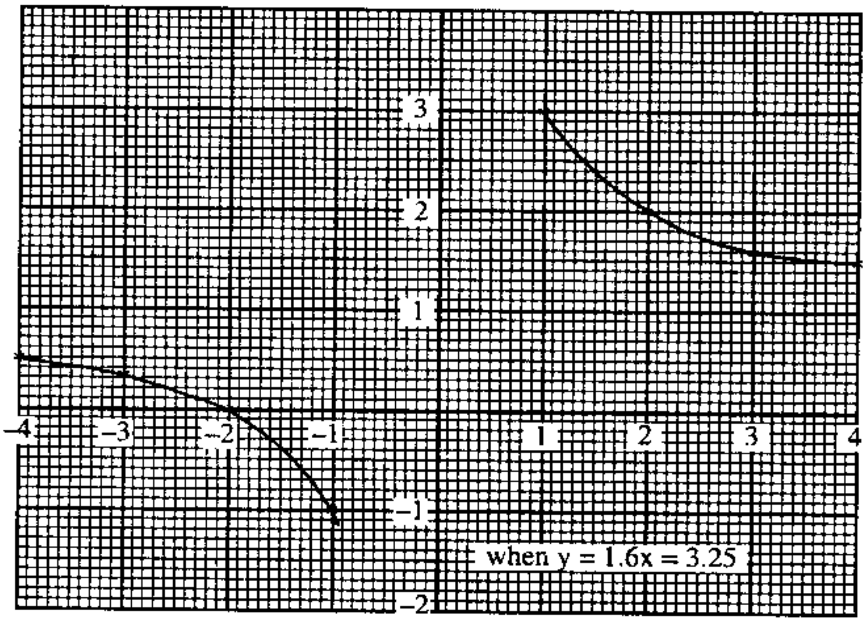


Fig. 15.2

Note that x = 0 is an asymptote
When y = 1.6, x = 3.25

Exercise 15.1

1. Copy and complete table 15.3 below for the equation, $y = 2^x + 1$. Hence draw the graph.

Table 15.3

x	0	1	2	3	4	5
2^x	1			8		32
y	2			—		33

2. Draw the graph of $y = 2^{x-1} - 1$ for $-2 \leq x \leq 5$. Use the graph to find the value of y when x = 0.5 and x = 2.5.

3. Draw the graph of $y = 4 + \frac{3}{x}$ for $-3 \leq x \leq 3$. From the graph, find
   (a) y when x = 1.5.
   (b) x when y = 5.4.

4. Draw the graph of $y = \frac{3x}{x+3}$ for values of x from –2 to 6.
5. Draw the graph of $y = 5 - \frac{2}{3}x$.
6. Draw the graph of $y = \frac{x^2}{3} - 4$.

15.2: Graphical Solution of Cubic Equations

An expression of the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$ is called a **cubic expression**.

The following are examples of cubic expressions:

(a) $4x^3 + 2x^2 + 5x + 1$ (b) $x^3 - 7x^2 + 6$ (c) $2x^3 + 4$ (d) $-x^3$

In a cubic expressions, the highest power of the variable is 3.

Consider the equation $y = 2x^3 - 3x^2 - 3x + 2$. Table 15.4 gives the corresponding values of x and y.

Table 15.4

x	–3	–2	–1	0	1	2	3	4
$2x^3$	–54	–16	–2	0	2	16	54	128
$-3x^2$	–27	–12	–3	0	–3	–12	–27	–48
$-3x$	9	6	3	0	–3	–6	–9	–12
+2	2	2	2	2	2	2	2	2
$y = 2x^3 - 3x^2 + 2$	–70	–26	0	2	–2	0	20	70

When we plot these points, we get the curve shown in figure 15.3. We can use the graph to solve the equation $2x^3 - 3x^2 - 3x + 2 = 0$.

The solution is given by reading off the values of x where the graph cuts the line y = 0 (the x-axis). In this case, the values of x are –1, $\frac{1}{2}$ and 2.

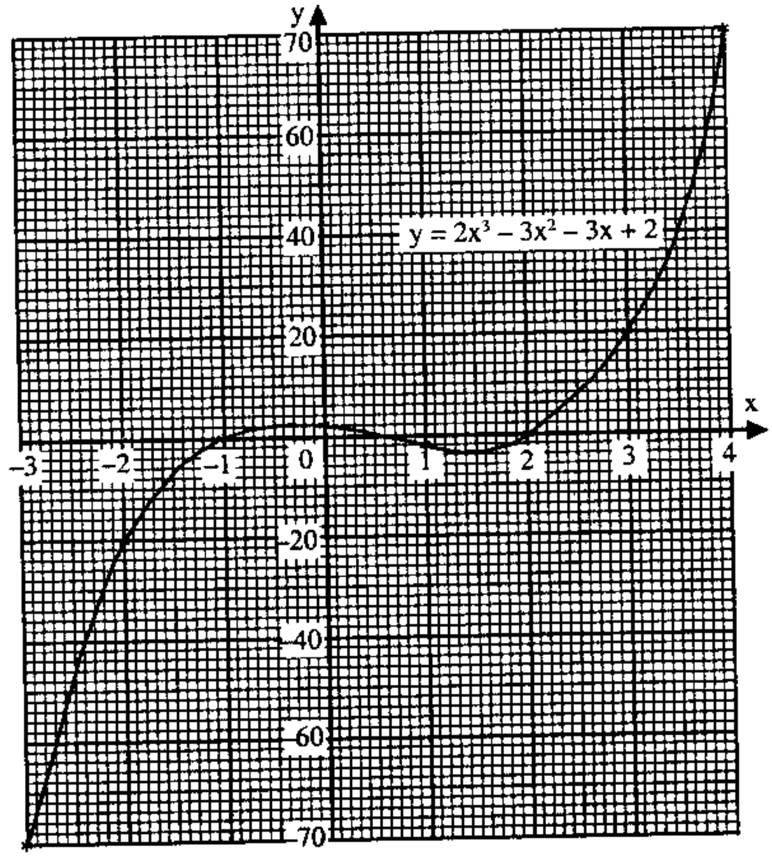


Fig. 15.3

Example 3

Draw the graph of $y = 2x^3 + x^2 + 2$ for the interval $-3 \leq x \leq 3$.

(a) Use your graph to solve the equation $2x^3 + x^2 - 5x + 2 = 0$.

(b) By drawing a suitable straight line using the same axes, solve the equation $2x^3 + x^2 - 5x + 2 = 6x + 12$.

Solution

(a) Table 15.5 shows the corresponding values of x and y for $y = 2x^3 + x^2 - 5x + 2$.

Table 15.5

x	–3	–2	–1	0	1	2	3
$2x^3$	–54	–16	–2	0	2	16	54
x^2	9	4	1	0	1	4	9
$-5x$	15	10	5	0	–5	–10	–15
+2	2	2	2	2	2	2	2
$y = 2x^3 + x^2 - 5x + 2$	–28	0	6	2	0	12	50

From the table, we obtain the curve shown in figure 15.4.

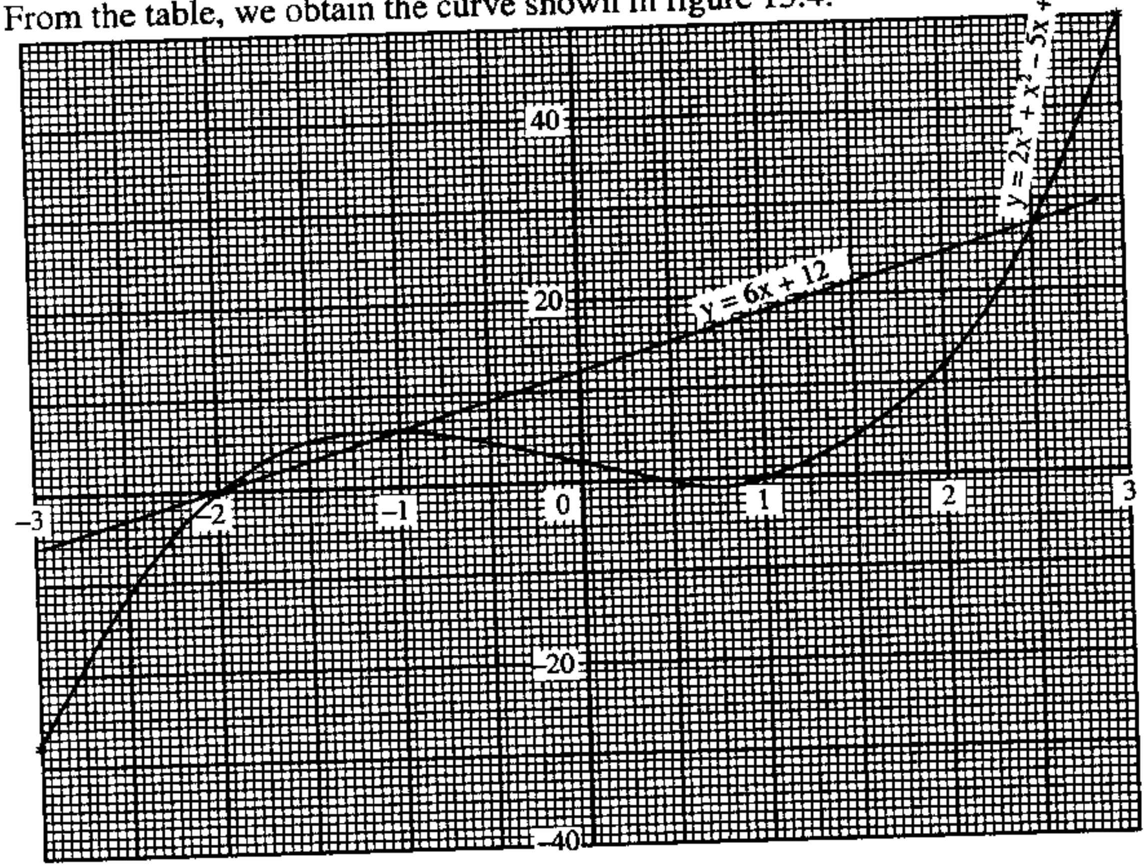


Fig. 15.4

From the graph the solutions of the equation

$2x^3 + x^2 - 5x + 2 = 0$ are $x = -2$, $x = \frac{1}{2}$ and $x = 1$.

(ii) To solve the equation $2x^3 + x^2 - 5x + 2 = 6x + 12$, we draw the straight line $y = 6x + 12$ (see figure 15.4). We then read off the co-ordinates of the points of intersection of the curve and the straight line. The points are (–2, 0), (–1, 6) and (2.5, 27). Therefore, the solutions of the equation $2x^3 + x^2 - 5x + 12 = 6x + 12$ are $x = -2$, $x = -1$ and $x = 2.5$.

Exercise 15.2

1. Copy and complete tabie 14.3 for the equation $y = 5x^3 + 2x^2 - 5x - 2$

Table 15.4

x	–2	–1	0	1	2
$-5x$	10	5	0	–5	–10
$2x^2$	8		0		
$5x^3$			0		40
y	–24				

By drawing the graph of $y = 5x^3 + 2x^2 - 5x - 2$, solve the equation $5x^3 + 2x^2 - 5x - 2 = 0$.

2. Draw the graph $y = x^3 - 2x^2 - 9x + 8$ for $-4 \leq x \leq 5$. Use the graph to solve the equation $x^3 - 2x^2 - 9x + 8 = 0$
3. Draw the graph of $y = x^3 - 4x + 12$ for the interval $-4 \leq x \leq 3$. By drawing a suitable straight line, determine the values of x and y for which

   $x^3 - 4x + 12 = 12 - \frac{15x}{4}$
4. Draw on the same axes the graphs of $y = x^3 + 2x$ and $y = x^2 - 4x + 2$ for the interval $-3 \leq x \leq 3$. Use your graphs to solve the equation $x^3 - x^2 + 6x - 2 = 0$ for the interval $-3 \leq x \leq 3$.
5. Copy and complete table 15.5 for $y = 2x^3 - 5x^2 + 5$.

Table 15.5

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2	2.5
$2x^3$		–6.75					2			
$-5x^2$				–1.25						–31.25
+5										
$y = 2x^3 - 5x^2 + 5$	31			3.5						

By drawing the graph of $y = 2x^3 - 5x^2 + 5$ and that of a suitable straight line on the same axes, determine the roots of equation $8x^3 - 20x^2 + 6x + 9 = 0$.

15.3: Average Rate of Change

Figure 15.5 is a distance-time graph for a motorist. The graph is a straight line. From the graph:

(a) What is the distance covered by the end of:
   (i) the first second?
   (ii) the third second?

(b) What is the speed during
   (i) the first second?
   (ii) the third second?

(c) What is the gradient of this straight line graph?

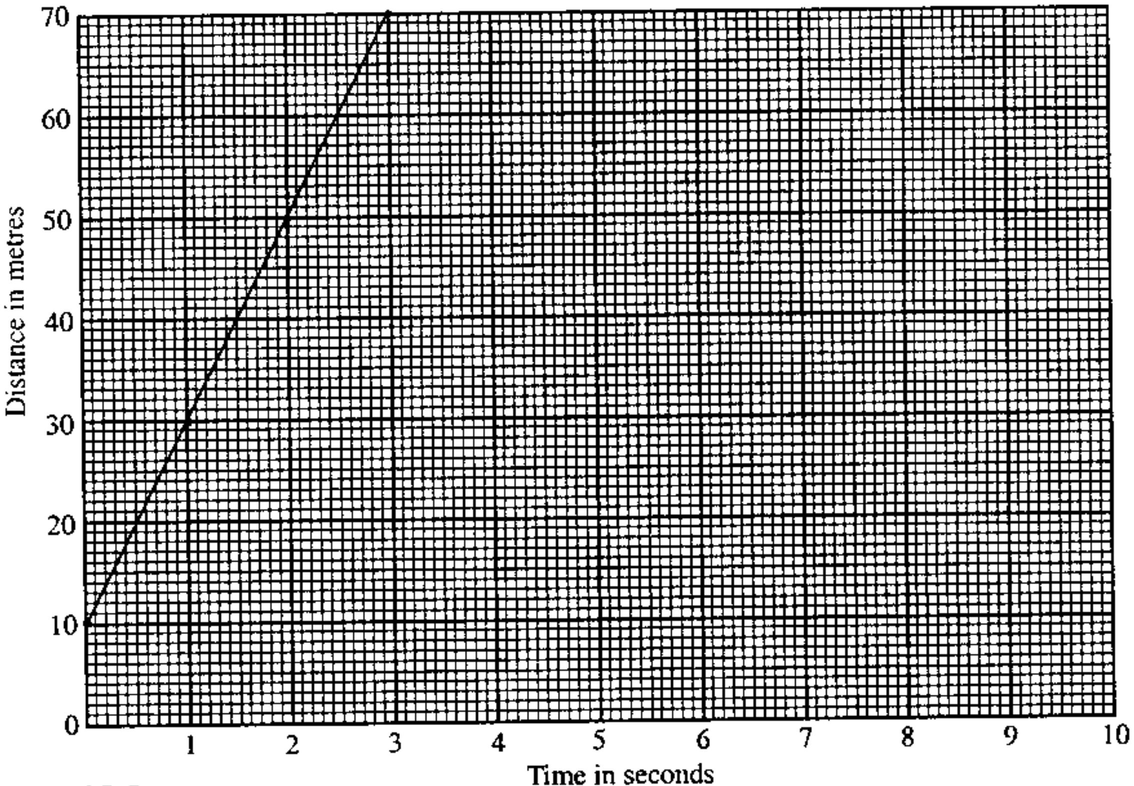


Fig. 15.5

We notice that the gradient of the straight line is 20, which is constant. The gradient represents the rate of change of distance with time (speed) which is 20 ms^{-1}. In many situations, the rate of change is not constant. Figure 15.6 is a graph representing the growth of a seedling.

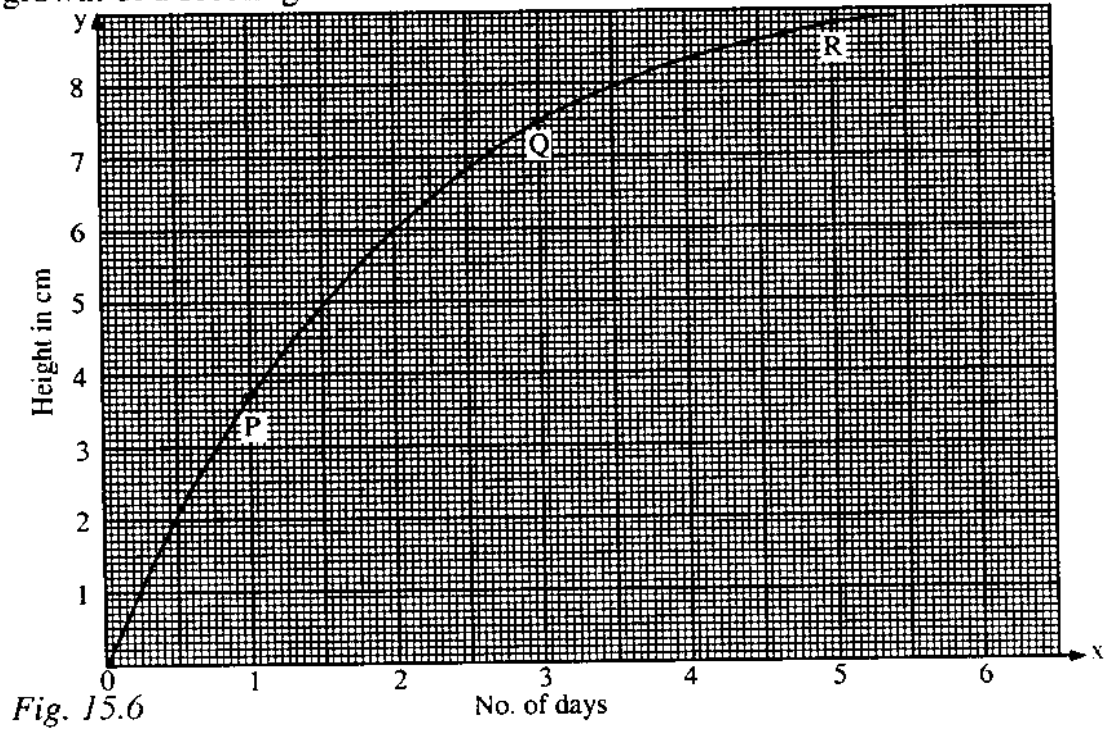


Fig. 15.6

From the graph, the change in height between day 1 and day 3 is given by; 7.5 cm – 3.8 cm = 3.7 cm. In these two days, the seedling gained a height of 3.7 cm.

The average rate of change of height is $\frac{3.7 \text{ cm}}{2 \text{ days}}$ = 1.85 cm/day

During the next two days, the gain in height is 8.8 cm – 7.5 cm = 1.3 cm

Therefore, the average rate of growth is $\frac{1.3 \text{ cm}}{2 \text{ days}}$ = 0.65 cm/day.

Notice that the rate of growth in the first 2 days was 1.85 cm/day while that in the next two days is only 0.65 cm/day. These rates of change are represented by the gradients of the lines PQ and QR respectively. From the graph, find:

(i) the average rate of growth between day 2 and day 4.
(ii) the average rate of growth between day 1 and day 5.
(iii) the average rate of growth from the beginning to day 5.

Exercise 15.3

1. The graph in figure 15.7 shows how the speed of a car varied.

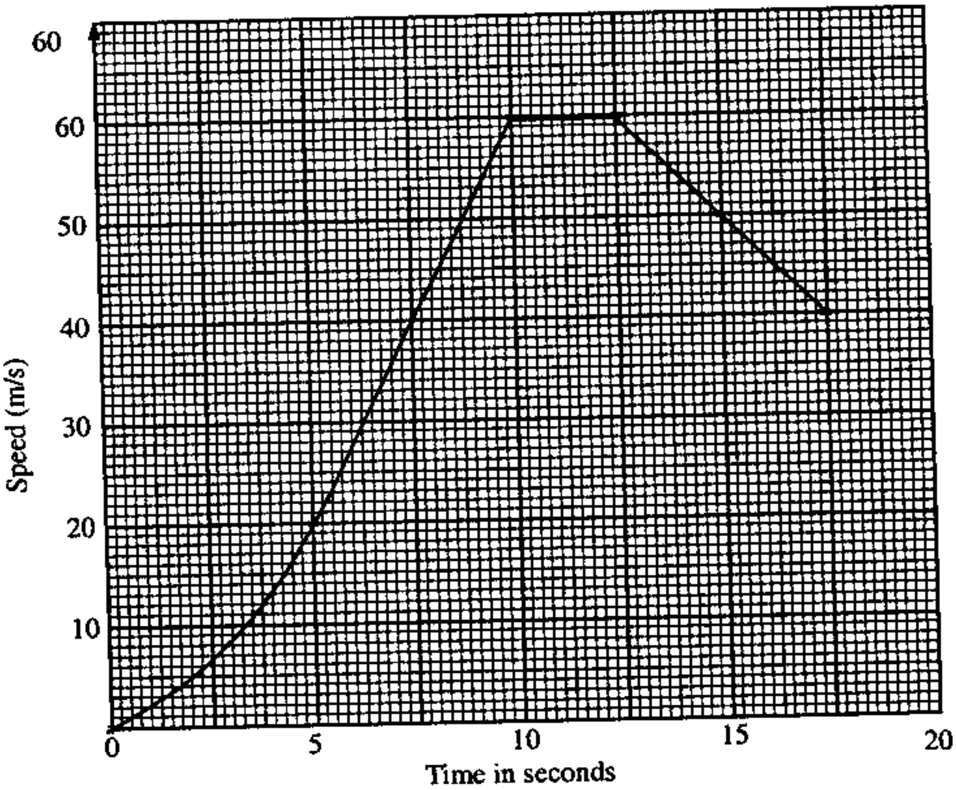


Fig. 15.7

Find the average change of speed:
(a) in the first five seconds.
(b) between 5th and 10th seconds.
(c) between t = 12.5 seconds and t = 17.5 seconds.

2. Table 15.8 shows how the temperature of a liquid being heated changed with time.

Table 15.8

Time (sec)	0	1	2	3	4	5	6
Temperature (°C)	1	2	5	10	17	26	27

Use this information to draw a graph of temperature against time. From your graph, find the average rate of change of temperature between the:
(a) 2nd and 4th seconds.
(b) 3rd and 5th seconds.

3. Table 15.9 below shows the population of a certain town between 1990 and 200.

Table 15.9

Year	1990	1992	1994	1996	1998	200
Population	70 000	71 000	73 500	76 500	81 000	86 500

(a) Use this information to draw a graph of population against time.
(b) From your graph, find:
  (i) the population of the town in 1997.
  (ii) the average rate of population growth between 1990 and 1994, 1996 and 1998.

4. The graph in figure 15.8 shows how the depth of water in a tank varied over a period of 14 days. Determine the average rate of change in depth:
(a) between L and M.
(b) between M and N.
(c) between the 1st and 5th day.
(d) for the last five days.

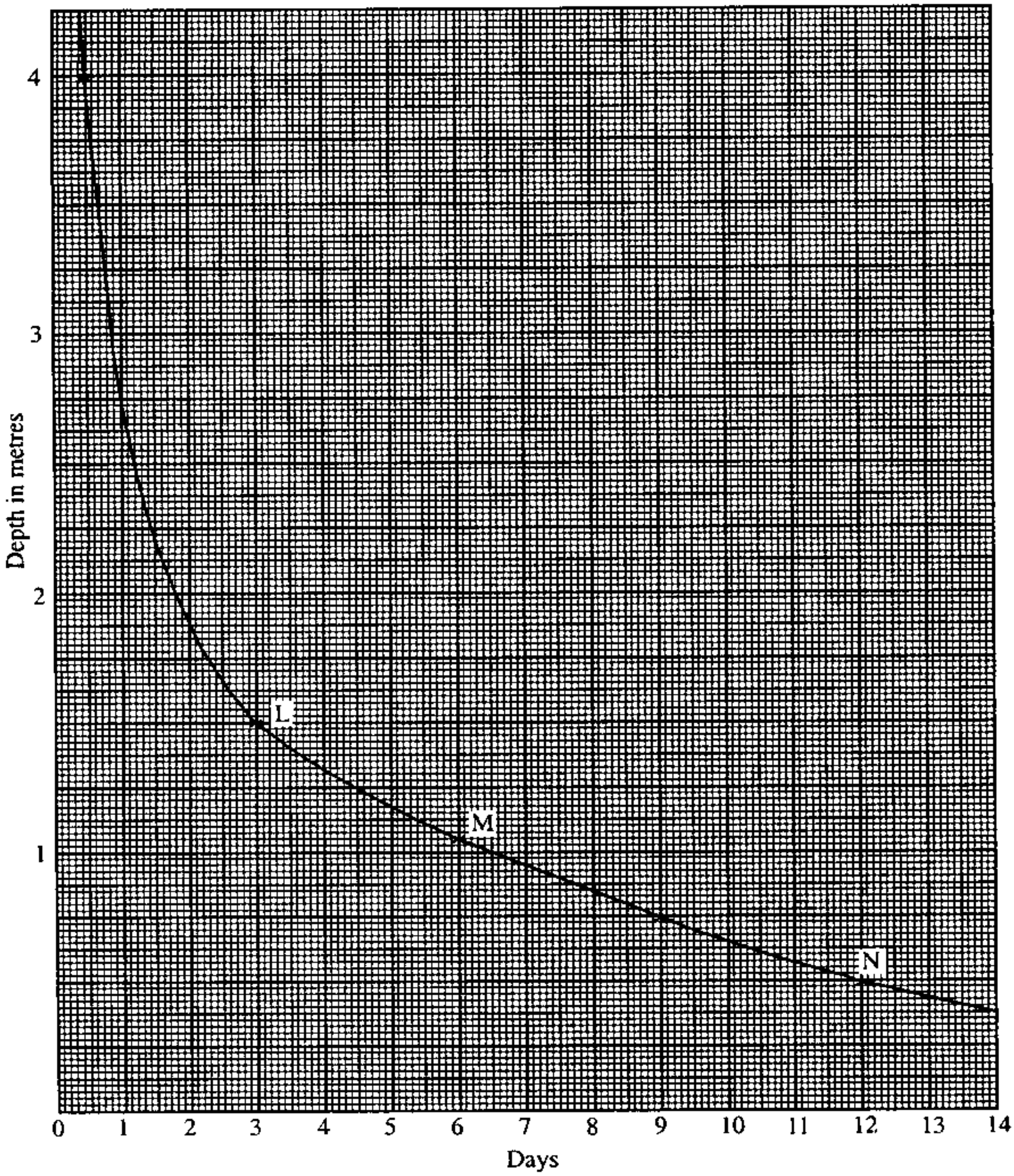


Fig. 15.8

5. The graph in figure 15.9 shows the number (N) of organisms in a colony. At the end of the fourth day, the colony was interfered with and it started to decay (decrease in number).

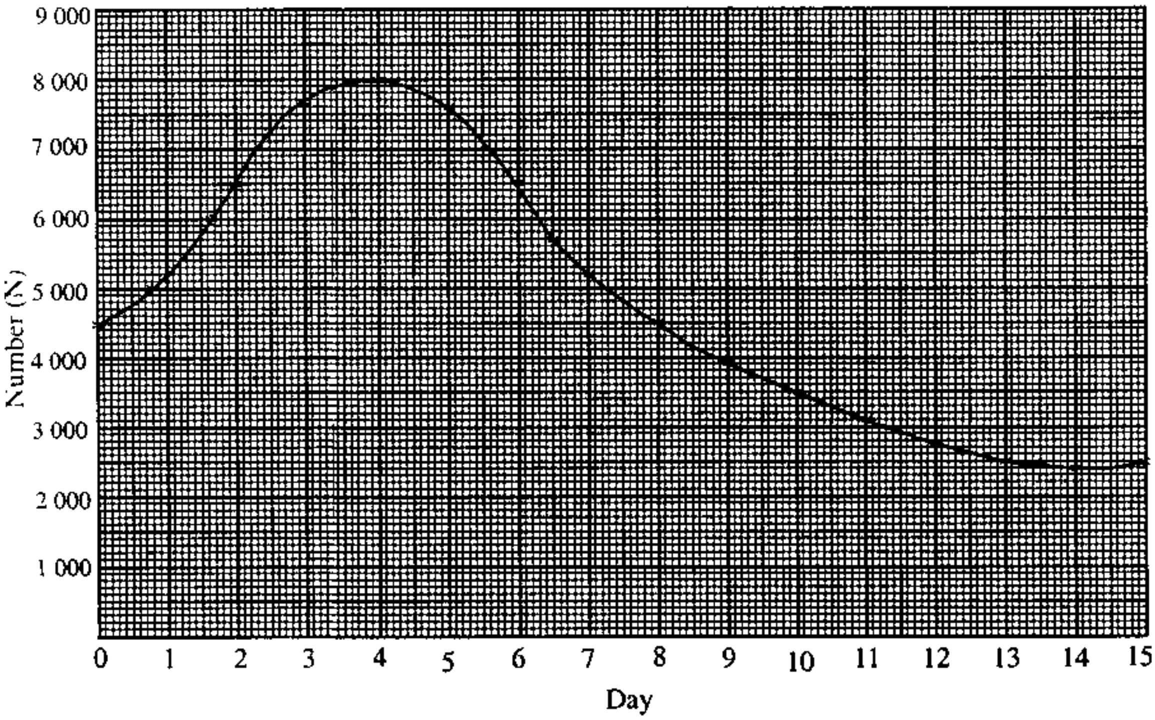


Fig. 15.9

Use the graph to estimate:

(a) the average rate of growth during the first four days.

(b) the average rate of decay during the next four days.

(c) a period of two days during which the average rate of change was zero.

Copy and complete table 15.10 and use it to study the trend in the rate of decay as time varies.

Table 15.10

	Average rate of decay
Day 7 to day 9	
Day 8 to day 10	
Day 9 to day 11	
Day 10 to day 12	
Day 11 to day 13	
Day 12 to day 14	
Day 13 to day 15	

5.4: Rate of Change at an Instant

In the last section, we learnt how to find the average rate of change during an interval.

In this section, we are going to learn how to determine the rate of change at a point (at an instant). Figure 15.10 shows the graph of the growth of the seedling which was considered in figure 15.6.

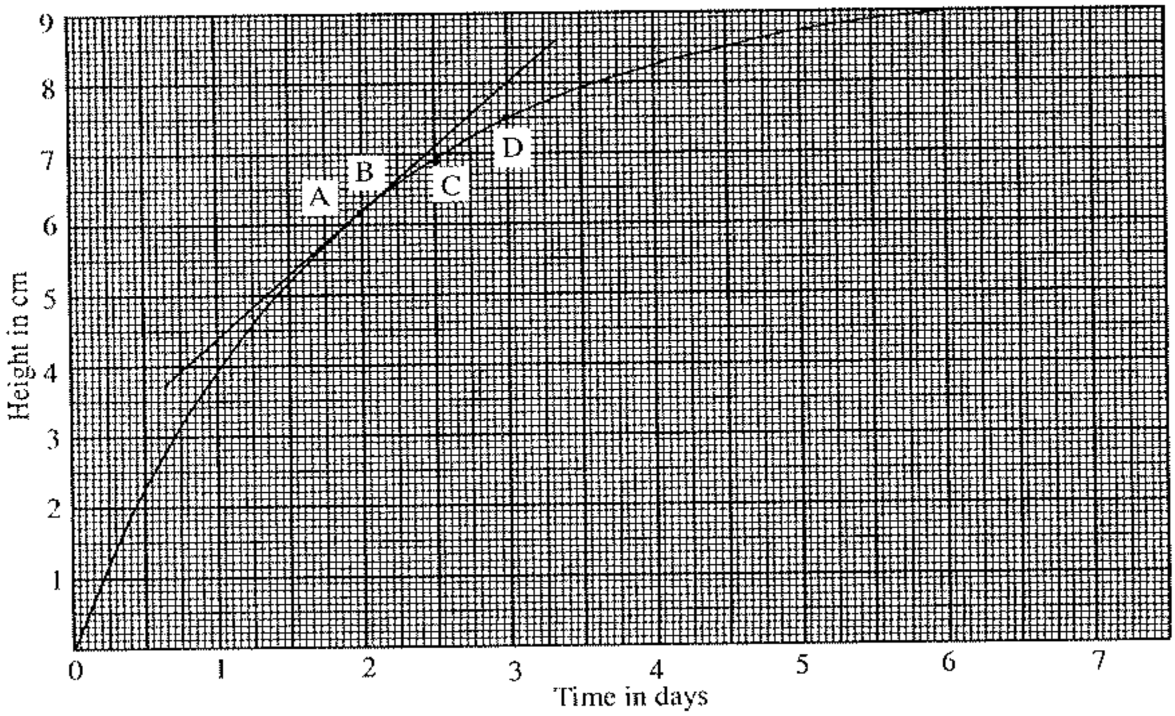


Fig. 15.10

Table 15.11

Interval	*Width of Interval*	*Gradient*
AF	2	$\frac{8.4-6.1}{2} = 1.15$
AE	1.5	$\frac{8.0-6.1}{1.5} = 1.27$
AD	1	$\frac{7.5-6.1}{1} = 1.40$
AC	0.5	$\frac{6.9-6.1}{0.5} = 1.60$
AB	0.25	$\frac{6.5-6.1}{0.25} = 1.60$

We can keep reducing the time intervals towards t = 2. As we reduce the time intervals, the gradients come closer and closer to the gradients of the tangent at A.

The gradient of the tangent at point A gives the rate of growth at that particular point.

From the graph, the gradient of the tangent at A is 1.7.

Using the graph in figure 15.10 determine the rate of growth at:

(i) t = 1 (ii) t = 3 (iii) t = 5 (iv) t = 3.5

Note:

We have seen that to find the rate of change at an instant (particular point), we:

(i) draw a tangent to the curve at that point.

(ii) determine the gradient of the tangent.

Generally, the tangent of a curve at a point is the rate of change at that point. This is equal to the gradient of the tangent to the curve at the point.

Example 4

Draw the graph of $y = 5x^3 - 15x + 7$ for $-2 \leq x \leq 2$ and find the gradient to the curve at the following points:

(a) $x = 1.25$

(b) $x = 1$

(c) $x = -0.5$

Solution

x	–2	–1	0	1	2
$5x^3$	–40	–5	0	5	40
$-15x$	30	15	0	–15	–30
+7	7	7	7	7	7
$y = 5x^3 - 5x + 7$	–3	17	7	–3	17

Figure 15.11 shows the graph of the curve $y = 5x^3 - 15x + 7$ with tangents drawn at the points x = 1.25, x = 1 and x = 0.5. From the graph, the gradient at:

(a) $x = 1.25$ is $\frac{20 - 5}{3.7 - 2} = \frac{15}{1.7}$

$= 8.8$

(b) $x = 1$ is 0 (since tangent is horizontal).

(c) $x = -0.5$ is $\frac{20 - 2.5}{-1 - 0.5} = \frac{17.5}{-1.5}$

$= 11.7$

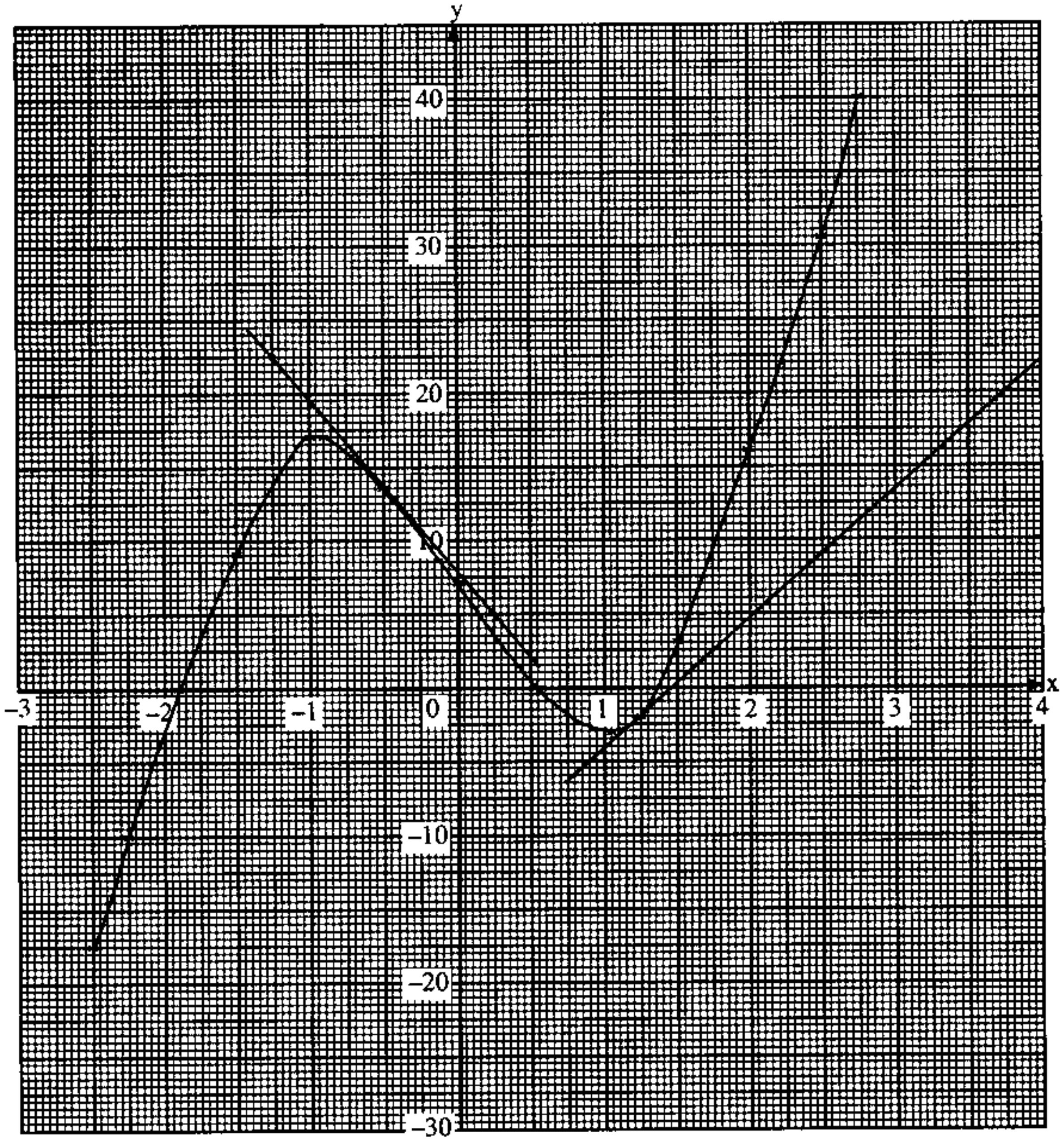


Fig. 15.11

Exercise 15.4

1. Draw the graph of $y = x^2 - 5x$ for $-5 \leq x \leq 5$.
   Use your graph to find the gradient of the curve at the following points:
   (a) $x = 1.5$ (b) $x = 2.5$ (c) $x = -2.5$
2. Figure 15.12 is a graph whose equation is $y^2 = 36 - x^2$ for $0 \leq x \leq 6$ and $y \geq 0$. Use it to find the gradient at:
   (a) $x = 2$ (b) $x = 4.3$

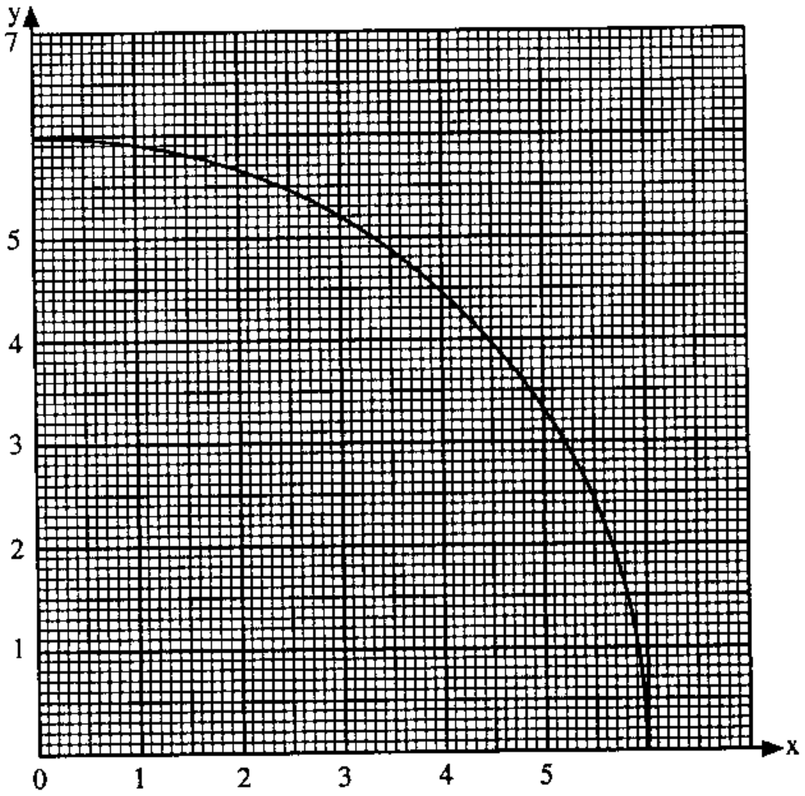


Fig. 15.12

3. The distance s(cm) covered by a toy-car at intervals of one second are given in table 15.12 below.

Table 15.12

t (secs)	1	2	3	4	5	6
s (cm)	2	10	30	68	130	222

Draw the graph and use it to find:

(a) the speed when t = 3.5.

(b) the average speed between the 3rd and 5th seconds.

4. The volume of an elastic bag is increasing as indicated in table 15.13.

Table 15.13

t(sec)	0	1	2	3	4	5	6	7	8
v(cm^3)	5	5.2	5.6	10.4	17.8	30	48.2	73.6	107.4

Draw the graph and find the rate at which the volume is changing when:

(a) t = 4.4 (b) t = 7.3

(c) Find also the average rate of change of volume between the 4th and 7th seconds.

5. Draw the graph of $y = 2x^2$ for $0 \leq x \leq 8$. Use your graph to find the gradient at:
   (a) $x = 3.7$.
   (b) $x = 6.9$.
6. The graph in figure 15.13 shows how the velocity of a certain vehicle varied with time:

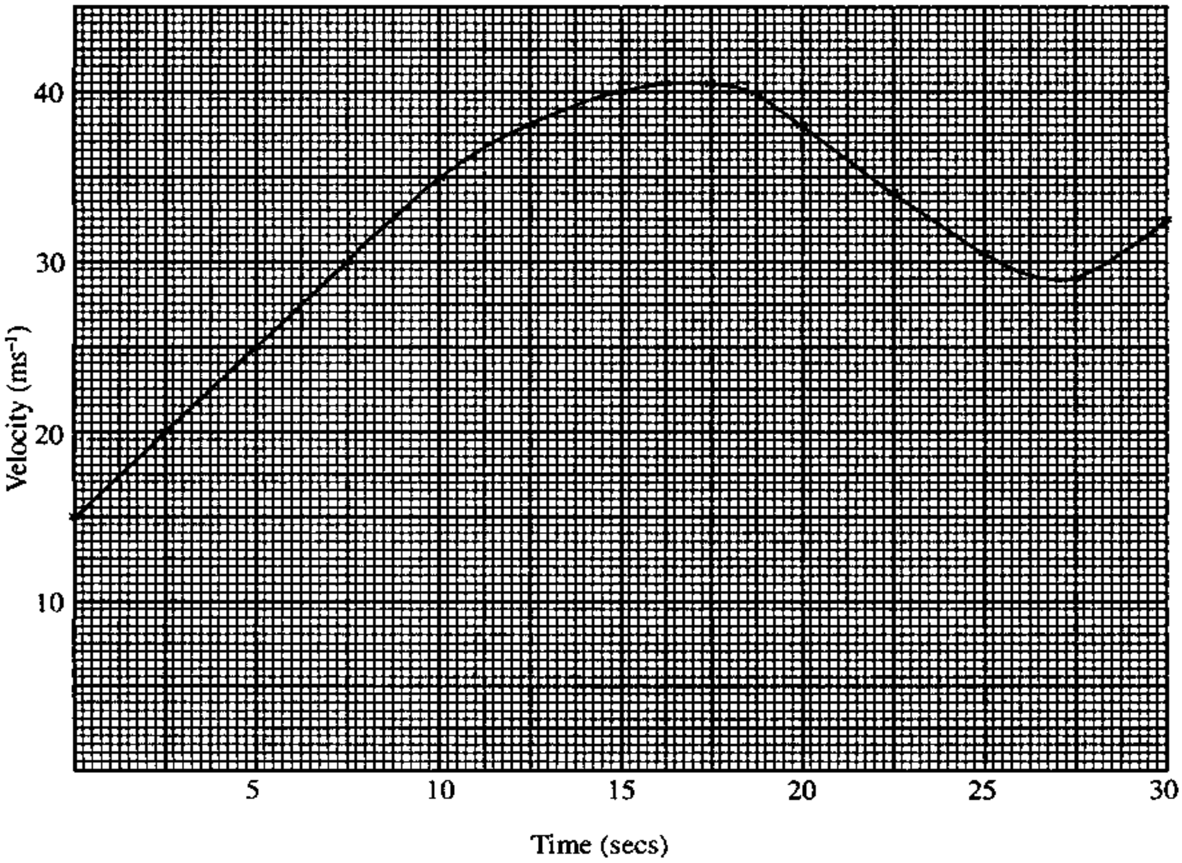


Fig. 15.13

By drawing suitable tangents, determine the rate of change of velocity when;
(a) $t = 9$ sec (b) $t = 20$ sec (c)' $t = 27.5$ sec

7. Draw the graph of $y = x^3 - 2x^2 - 4$ for $-4 \leq x \leq 4$. Use the graph to find the gradient of the curve at $x = -2$ and $x = 2.5$.
8. The height h (m) of an object above the ground is given by the formula

h = 2t – t², where t is time in seconds. Draw the graph of height against time for $0 \le x \le 2$.

(a) What is the rate of change of height at t = 0.38 and t = 1.5?

(b) At what time is the gradient 0? Why do you think the gradient is 0 at this time?

9. Draw the graph of $y = \frac{12}{x}$ for $1 \le x \le 12$. Use your graph to find the gradient at x = 3. Hence, find the equation of the tangent to the curve at x = 3.

15.5: Empirical Graphs

Raw data collected from experimental observations normally have errors. Below is a table of results obtained from an experiment. The results show how length *l* (cm) of a metal rod varies with increase in temperature T (⁰C).

Table 15.14

T (°C)	0	1	2	3	4	5	6	7	8
l (cm)	4.0	4.3	4.7	4.9	5.0	5.5	5.9	6.0	6.4

Figure 15.14 shows the points plotted.

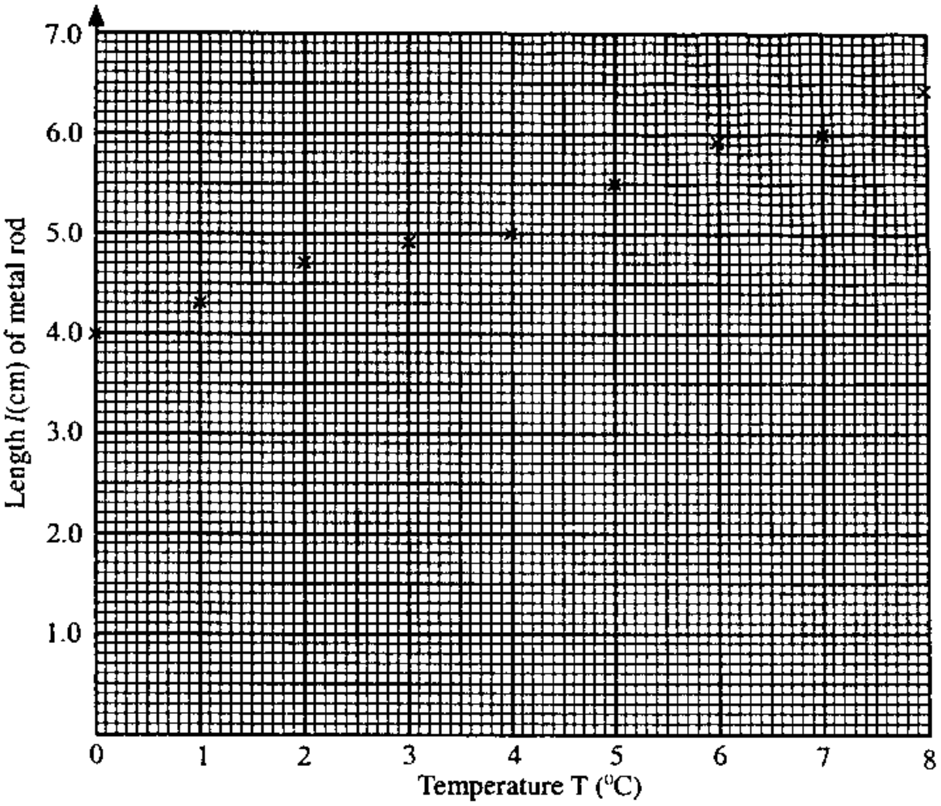


Fig. 15.14

The pattern that emerges after plotting the points suggest a linear relation between *l* and T. We therefore draw the best line possible through the points, as shown in figure 15.15.

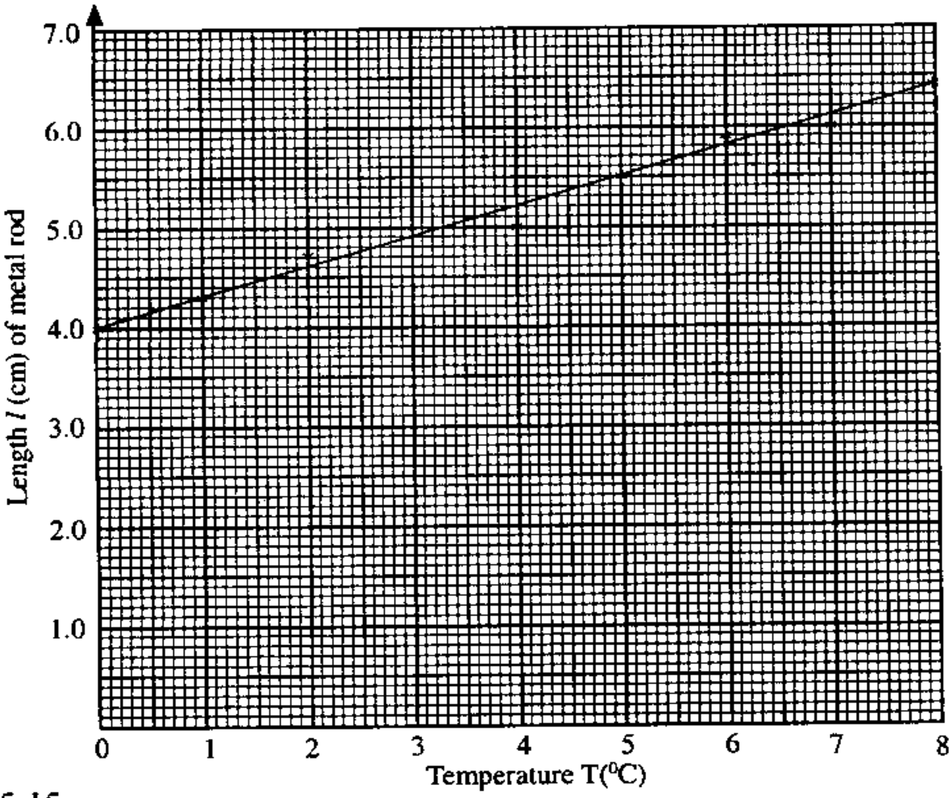


Fig. 15.15

This is called the line of best fit.

The line of best fit is drawn using a ruler (preferrably transparent) so that it passes through as many points as possible and the rest are evenly distributed below and above it.

This line cuts the y-axis at (0, 4) and passes through the point (5, 5.5). Therefore, the gradient of the line is $\frac{1.5}{5} = 0.3$. The equation of the line is;

$l = 0.3T + 4$

The gradient gives the increase in length per unit increase in temperature. The vertical intercept is the original length of the rod.

In general, linear laws take the form $y = mx + c$ where m and c are constants to be determined from the line of bes fit.

Exercise 15.5

1. Table 15.15 gives the mass and volume of a liquid measured at room temperature.

Table 15.15

m (g)	8	10	20	21	30	37	52	55	70
V (cm^3)	10	13	20	29	35	43	60	63	70

By plotting the points and drawing the line of best fit, estimate the density of the liquid. What is the equation of your graph?

2. Draw the line of best fit for the values of x and y given in table 15.16.

Table 15.16

x	1	1.5	2.5	2.9	3	3.5	4.3	4.9
y	7	8	10	10.9	12	11.0	13	14.8

What is the equation of your line?

3. For the values of x and y in table 15.17, plot the points and draw the line of best fit.

Table 15.17

x	30	33	38	45	48	55	60	68
y	0.1	0.15	0.25	0.27	0.35	0.5	0.55	0.7

4. A boy performed an experiment by immersing identical marbles in a certain amount of water contained in a measuring cylinder. Table 15.18 gives the results he obtained.

Table 15.18

No. of marbles	3	4	5	6	7	8	9
Readings (mm^3)	98.0	101.4	117.0	133.1	145.6	156.9	170.2

Plot the results on a graph paper and draw the line of best fit. What is the equation of the line you have drawn? What does:

(a) the gradient,

(b) the vertical intercept represent?

5. A girl used a string to measure circumference of cylinders without varying diameters.

Table 15.19 shows the results obtained.

Table 15.19

Diameter d (cm)	2	3	4	5	6	7	8	9
Circumference C (cm)	6.3	9.5	12.7	15.5	17.5	22.0	25.2	27.9

Use the results to draw a straight line graph. From your graph, estimate the value of π.

6. Telephone charges are worked out using the formula $s = mn + c$, where m is the amount charged in shillings per unit call, c the standing charge, n is the number of calls and s the total charge. Table 15.20 below which shows the number of calls made and the amount of money paid, contains two errors.

Table 15.20

No. of calls (n)	6	8	10	12	14	16	18	20
Money paid (s) (sh)	82	86	90	94	100	102	106	110

Draw the graph of the amount of money paid against the number of calls. From your graph find, the:

(a) charge per call.
(b) standing charge.

15.6: Reduction of Non-linear Laws to Linear Form

In chapter 9, we saw that when we plot the graph of $xy = k$, we get a curve. But when we plot y against $\frac{1}{x}$, we get a straight line whose gradient is k. The same approach is used to obtain linear relations from non-linear relations of the form $y = kx^n$.

Example 5

Table 15.21 shows the relationship between A and r.

Table 15.21

r	1	2	3	4	5
A	3.1	12.6	28.3	50.3	78.5

It is suspected that the relation is of the form $A = kr^2$. By drawing a suitable graph, verify the law connecting A and r and determine the value of K.

Solution

If we plot A against r^2, we should get a straight line.
The corresponding values of A and r^2 are given in table 15.22.

Table 15.22

r	1	2	3	4	5
r^2	1	4	9	16	25
A	3.1	12.6	28.3	50.3	78.5

Figure 15.16 is the graph of A plotted against r^2.

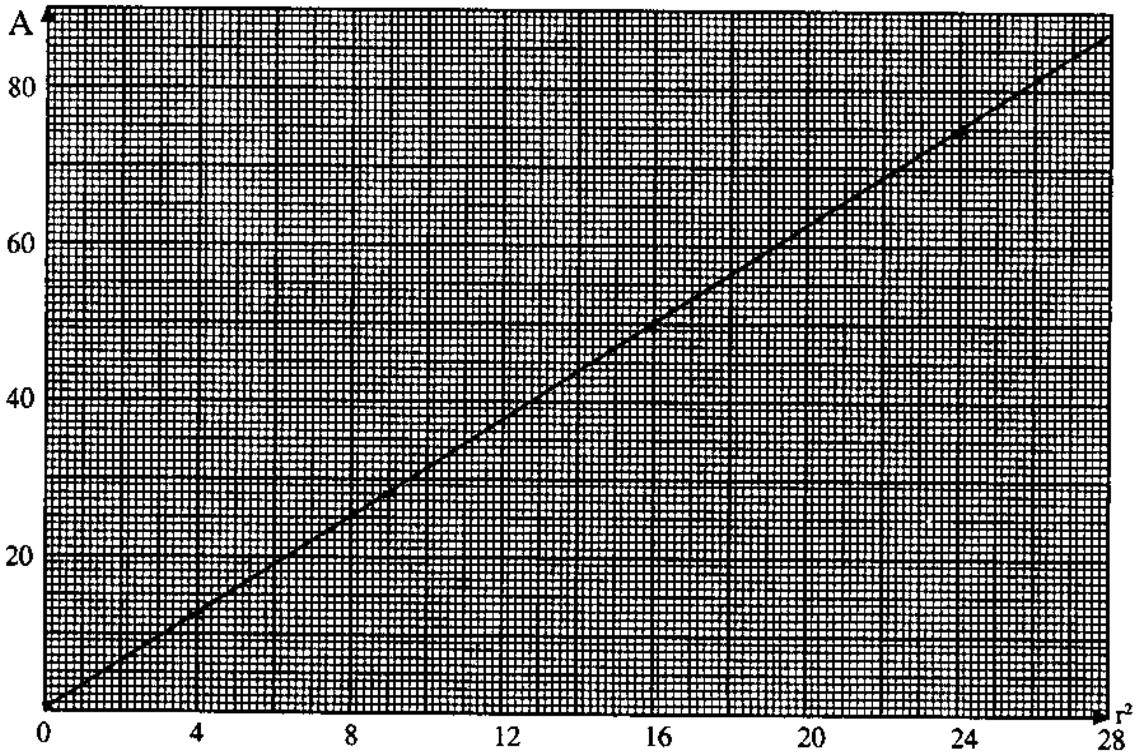


Fig. 15.16

Since the graph of A against r^2 is a straight line, the law $A = kr^2$ holds. The gradient of this line is 3.1 (1 d.p.). This is the value of k.

Example 6

From 1960 onwards, the population P of a certain town is believed to obey a law of the form $P = kA^t$, where k and A are constants and t is the time (in years) reckoned from 1960. Table 15.23 shows the population of the town since 1960.

Table 15.23

t	1960	1965	1970	1975	1980	1985	1990
P	5 000	6 080	7 400	9 010	10 960	13 330	16 200

By plotting a suitable graph, check whether the population growth obeys the given law. Use the graph to estimate the value of A.

Solution

The law to be tested is $P = kA^t$ Taking logs of both sides we get $\log P = \log (kA^t)$ $\log P = \log k + t \log A$, which is in the form $y = mx + C$. Thus we plot log P against t. (Note that log A is a constant).

Table 15.24 shows corresponding values of t and log P.

Table 15.24

t	1960	1965	1970	1975	1980	1985	1990
Log P	3.699	3.784	3.869	3.955	4.040	4.125	4.210

Figure 15.17 shows the graph of log P against t.

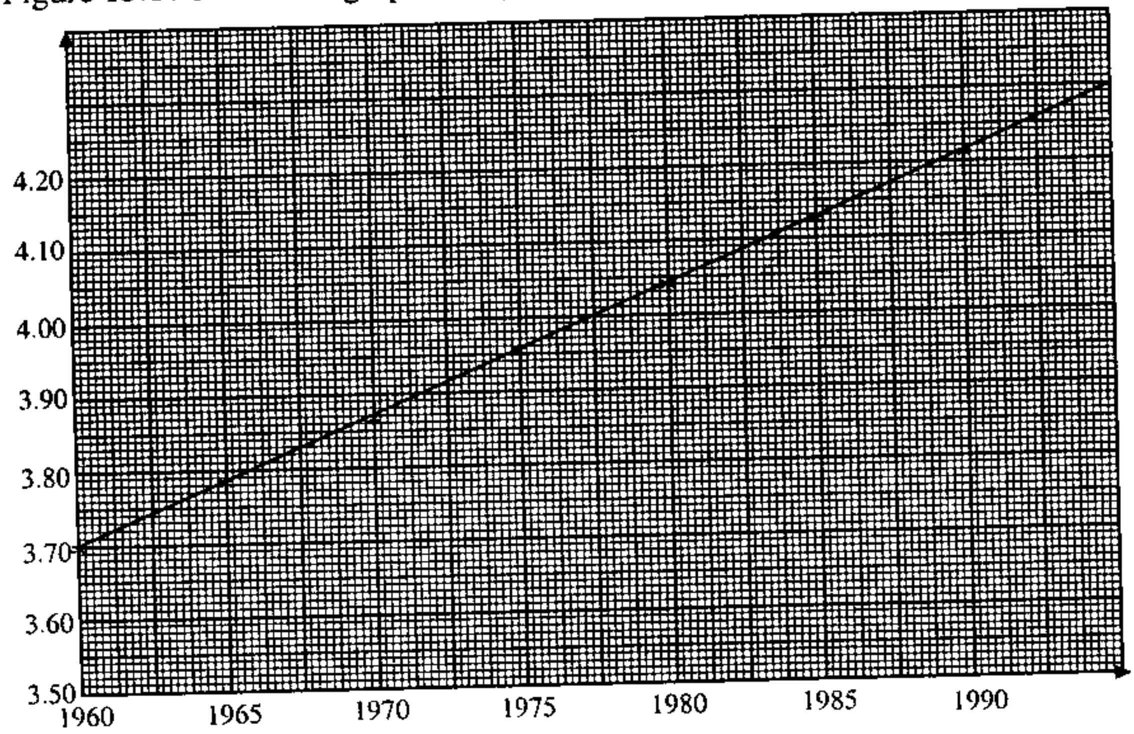


Fig. 15.17

Since the graph is a straight line, the law $P = kA^t$ holds.

Log A is given by the gradient of the straight line.

Therefore, $\log A = 0.017$

Hence, $A = 1.04$
log k is the vertical intercept.
Hence $\log k = 3.69$
Therefore $k = 4\,898$
Thus, the relationship is $P = 4\,898\,(1.04)^t$.

Generally, laws of the form $y = kA^x$ can be written in the linear form as; $\log y = \log k + x\log A$ (by taking logs of both sides).

When log y is plotted against x, a straight line is obtained. Its gradient is log A and the intercept is log k.

The law of the form $y = kx^n$, where k and n are constants can be written in linear form as;

$\log y = \log k + n \log x$. In this case, log y is plotted against log x. The gradient of the line gives n while the vertical intercept is log k.

Exercise 15.6

1. In each of the following, determine which quantities should be plotted against each other to give a straight line graph:
   - (a) $V = \frac{4}{3}\pi r^3$ (b) $P = \frac{k}{V}$ (k is a constant)
   - (c) $st^2 = 36$ (d) $r = at^3 + b$ (a and b are constants)
   - (e) $A = Pt^2 + k$ (P and k are constants).
2. By re-arranging the following where nècessary, state the quantities which should be plotted against each other to obtain a straight line:
   - (a) $xy = k + rx$ (k and r are constants)
   - (b) $uv = cu + d$ (c and d are constants)
   - (c) $t = s + kst$ (k is a constant)
   - (d) $p^2q - mq = c$ (m and c are constants)
   - (e) $y = mx^2 + n$ (m and n are constants)
3. The resistance R and the voltage V in a certain conductor are connected by the law $RV = a + bV$, where a and b are constants. What should you plot R against to get a straight line graph?
   - (a) Rewrite the equation in the form = mx + c.
   - (b) Use table 15.25 below and the equation obtained in (a) to draw a straight line graph.
   - (c) From your graph, estimate the values of a and b.

Table 15.25

R	1	2	3	4	5	6
V	0.25	0.13	0.08	0.06	0.05	0.04

4. The variables t and a are connected by the equation $t = \frac{a}{B} + P$, where B and P are constants.
   (a) Use table 15.26 below to draw a straight line graph.
   (b) From your graph estimate the values of B and P.

Table 15.26

t	1.1	1.2	1.3	1.4	1.5	1.6
a	–0.26	–0.22	–0.16	–0.11	–0.05	0.01

5. When a current of I amperes flows through a resistor, the power P watts dissipated is given by the relation $P = AI^n$, where A and n are constants. By taking logarithms, re-write the relation in the form $y = mx + c$. For the values of I and P in the table below, draw the graph of the linear relation.

Table 15.27

I	1	1.4	2	3	4.2	6.8
P	3	4.3	12	27	52.9	138.7

Use your graph to determine the values of A and n.

6. Two quantities are thought to be related by a law of the form $y = ax^n$, where a and n are constants. Some experimental values of x and y are shown in table 15.28.

Table 15.28

x	1	1.5	2.0	2.5	3.0	3.5
y	3.1	15.0	50.0	122.0	254.0	447.0

Use a suitable graph to estimate the values of a and n.

7. The relation between two variables E and F is believed to be of the form $F = a + bE^{-1}$, where a and b are constants. Table 15.29 shows corresponding values of E and F.

Table 15.29

E	1	2	3	4	5	6
F	7	6	5.7	5.5	5.4	5.3

By drawing a suitable graph, estimate the values of a and b.

8. The luminous intensity I of a lamp was measured for various values of voltage V across it. The results were as shown below:

Table 15.30

V(volts)	30	36	40	44	48	50	54
I (lux)	708	1 248	1 726	2 320	3 038	3 848	4 380

It is believed that V and I are related by an equation of the form $I = aV^n$ where a and n are constants.

(a) Draw a suitable linear graph and determine the values of a and n.

(b) From the graph find:

   (i) the value of I when V = 52.

   (ii) the value of V when I = 2 800

9. The voltage (V) between the plates of a charging condenser in a given time (t) is believed to obey a law of the form $V = Ak^t$, where A and k are constants. Table 15.31 below shows various values of voltage and time.

Table 15.31

t	0.4	0.8	1.2	1.6	2.0	2.4
v	48.8	65.0	72.8	89.0	108.8	132.8

One reading was mis-recorded. Draw a linear graph to verify the given law.

(a) Identify the mis-recorded reading from the graph and give the correct value.

(b) Find A from your graph.

(c) Find K from the graph.

10. Table 15.32 shows values obtained from an experiment.

Table 15.32

x	1	2	3	4	5	6
y	2.70	5.70	11.15	22.62	45.2	90.51

It is thought that y and x are connected by the formula;

$Y = A^{(b+x)}$

(a) By drawing a suitable linear graph, find the values of the constants A and b.

(b) From the graph, estimate:

   (i) the value of x when y = 24.

   (ii) the value of y when x = 5.6.

11. In an experiment, two quantities x and y were observed and the results obtained recorded as given in table 15.33 below.

Table 15.33

x	0	4	8	12	16	20
y	1.0	0.64	0.5	0.42	0.34	0.28

(a) By plotting $\frac{1}{y}$ against x, confirm that y is related to x by an equation of the form $y = \frac{q}{p + x}$ where p and q are constants.

(b) Use your graph to determine the values of p and q.

(c) Etimate the value of:

(i) y when x = 14.

(ii) x when y = 0.46.

12. The current I and voltage V in a circuit are thought to obey the law $I = kV^n$. Table 15.34 shows the corresponding values of I and V obtained in an experiment.

Table 15.34

I	1.7	3.5	7.2	12.1	18.1	25.1	33
V	0.8	1.2	1.8	2.4	3.0	3.6	4.2

By plotting a suitable linear graph, check whether the law is obeyed. Use your graph to estimate the values of k and n where k and n are constants.

13. The relation between the variables Y and X is conjectured to be $y = AB^x + 9.1$, where A and B are constants. Table 15.35 shows corresponding values of x and y.
By drawing a suitable straight line graph, estimate the values of A and B.

Table 15.35

x	1.4	2.3	3.2	4.0	5.0	6.1
y	9.5	10.0	12.6	17.5	33.2	90.4

15.7: Equation of a Circle

Figure 15.18 shows a circle centre (0, 0) and radius 3 units.

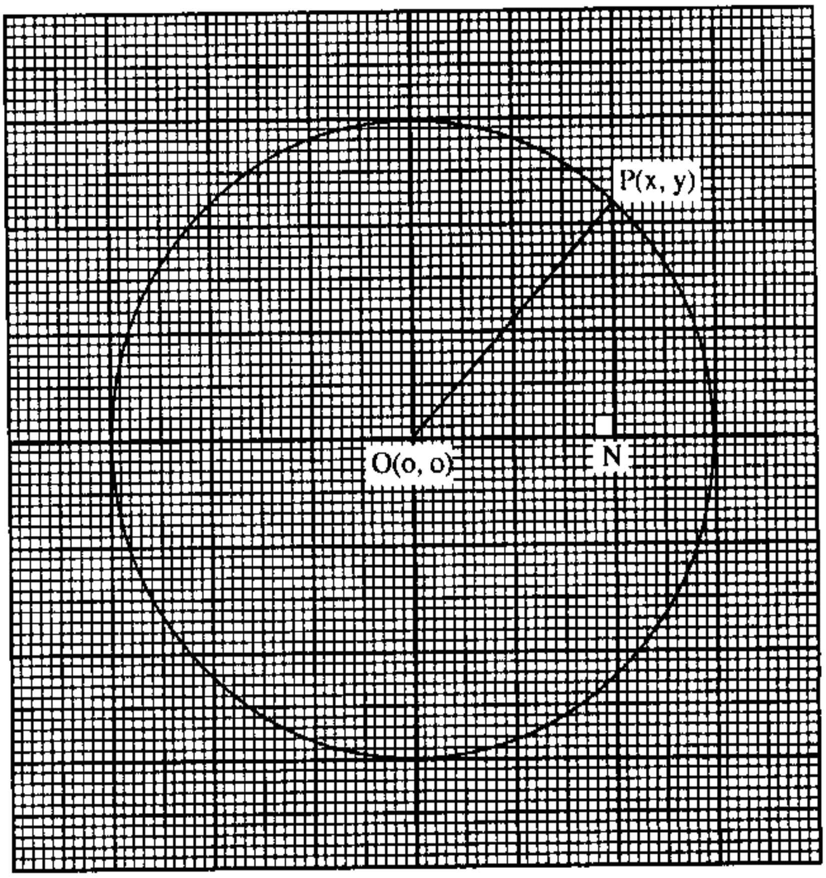


Fig. 15.18

P (x, y) is a point on the circle. Triangle PON is right-angled at N.

By Pythagoras' theorem;

$ON^2 + PN^2 = OP^2$

But ON = x, PN = y and OP = 3. Therefore, $x^2 + y^2 = 3^2$

This is the equation of a circle centre (0, 0) and radius 3 units.

In general, the equation is of a circle whose centre is (0, 0) and radius r is $x^2 + y^2 = r^2$.

Find the equation of the following circles.

(i) Centre (0, 0) radius 5 units.

(ii) Centre (0, 0) radius 6 units.

(iii) Centre (0, 0) radius $\sqrt{7}$ units.

Example 7

Find the equation of a circle centre (0, 0) and passing through (3, 4).

Solution

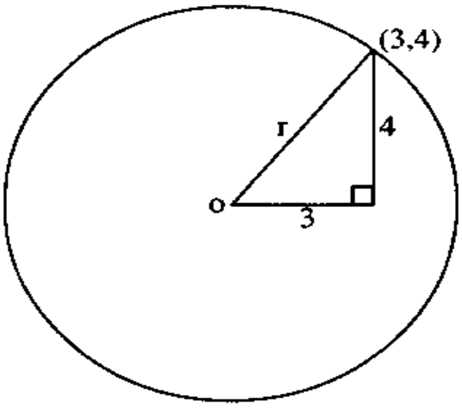


Fig. 15.19

Let the radius of the circle be r.
From Pythagoras' theorem;

$r = \sqrt{3^2 + 4^2}$
$r = 5$
$\therefore$ Equation of the circle is $x^2 + y^2 = 25$.

Consider a circle centre (5, 4) and radius 3 units.

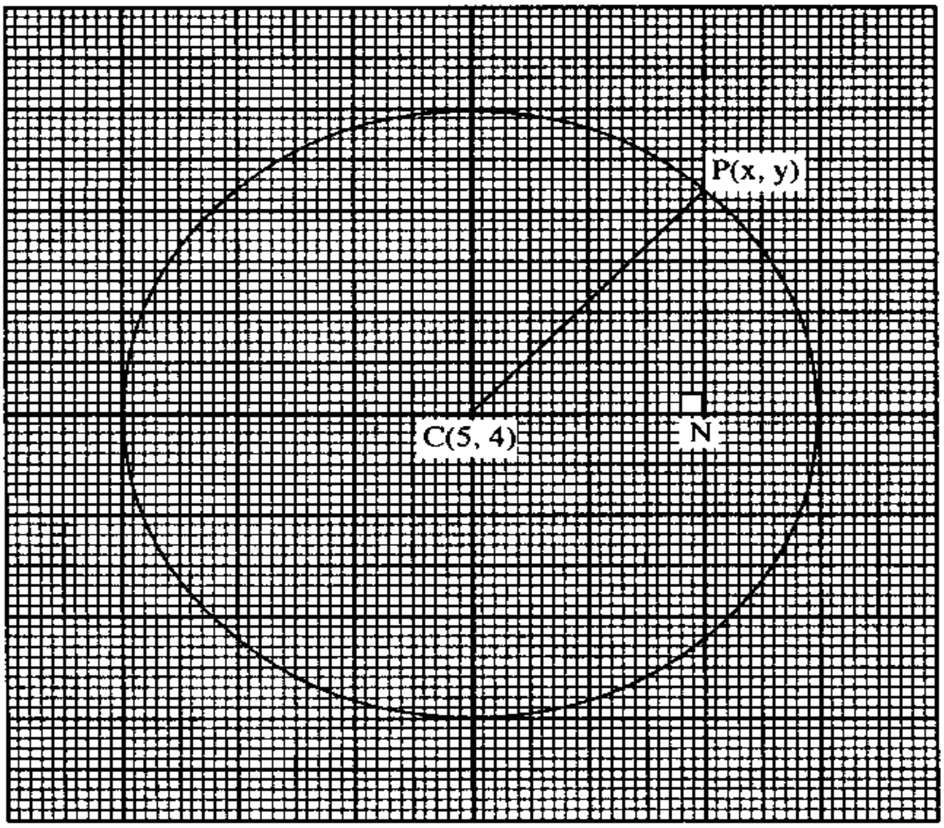


Fig. 15.20

In the figure, Δ CNP is right angled at N. By Pythagoras' theorem;
$CN^2 + NP^2 = CP^2$
But CN = (x – 5), NP = (y – 4) and CP = 3 units.
Therefore, $(x - 5)^2 + (y - 4)^2 = 3^2$. This is the equation of the circle.
In general, the equation of a circle centre (a, b) and radius r units is given by:
$(x - a)^2 + (y - b)^2 = r^2$

Example 8

Find the equation of a circle centre (–2, 3) and radius 4 units.

Solution

General equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$.
In the above case, a = 2, b = 3 and r = 4 units.
∴ Equation of the circle is $\{x - (-2)\}^2 + (y - 3)^2 = 4^2$, i.e.,
$(x + 2)^2 + (y - 3)^2 = 16$

Example 9

Line AB is the diameter of a circle such that the co-ordinates of A and B are (–1, 1) and (5, 1) respectively.

(a) Determine the centre and the radius of the circle.
(b) Hence, find the equation of the circle.

Solution

(a) Centre is the midpoint of AB.

$$\left(\frac{-1+5}{2}, \frac{1+1}{2}\right) = (2, 1)$$

$$\begin{aligned} \text{Radius} &= \sqrt{(5-2)^2 + (1-1)^2} \\ &= \sqrt{3^2} \\ &= 3 \end{aligned}$$

(b) Equation of the circle is;
$(x - 2)^2 + (y - 1)^2 = 3^2$
∴ $(x - 2)^2 + (y - 1)^2 = 9$.

Example 10

The equation of a circle is given by $x^2 - 6x + y^2 + 4y - 3 = 0$. Determine the centre and the radius of the circle.

Solution

$x^2 - 6x + y^2 + 4y$
$(x - 3)^2 + (y + 2)^2 = 16$

$x^2 - 6x + y^2 + 4y = 3$

Completing the square on L.H.S.;

$x^2 - 6x + 9 + y^2 + 4y + 4 = 3 + 9 + 4$

$(x - 3)^2 + (y + 2)^2 = 4 - 3 = 0$

∴ Centre of the circle is (3, –2) and radius is 4 units.

Alternatively;

$(x - a)^2 + (y - b)^2 = r^2$.

Expanding the expressions on the L.H.S.;

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 = 0$ .... (1)

Given equation: $x^2 - 6x + y^2 + 4y - 3 = 0$ .... (2)

Comparing coefficients;

$-2a = -6 \Rightarrow a = 3$

$-2 = 4 \Rightarrow b = -2$

$a^2 + b^2 - r^2 = -3$

$3^2 + (-2)^2 - r^2 = -3.$

$9 + 4 - r^2 = -3.$

$r^2 = 16$

$r = 4$

∴ Centre of the circle is (3, –2) and radius is 4.

Exercise 15.7

1. Find the equation of a circle centre:
   (a) (0, 0) and radius 2.
   (b) (0, 0) and radius $\sqrt{5}$.
   (c) (4, –1) and radius 3.
   (d) (–1, 3) and radius 4.
   (e) (–a, 1) and radius C.
2. Determine the equation of a circle given the co-ordinates of the centre and a point on the circle.
   (a) Centre (2, 0), passing through (2, 3)
   (b) Centre (2, 5), passing through (5, 1)
   (c) Centre (–2, 1), passing through (0, 4)
   (d) Centre (–3, –2), passing through (–6, –7)
3. Line AB is the diameter of a circle. Find the equation of the circle, given the co-ordinates of A and B as:
   (a) (–1, –4) , (7, 2)
   (b) (0, 4) , (4, 7)
   (c) (1, 7) , (11, 7)

4. The equation of a circle is given by $x^2 + 8x + y^2 - 2y - 1 = 0$. Determine the radius and centre of the circle.
5. Show that the equation of a circle with centre at $(-3, 3)$ and radius 5 units is given by: $x^2 + y^2 + 6x - 6y = 7$.
6. Find the centre and radius of a circle whose equation is: $x^2 + 4x + y^2 - 5 = 0$.
7. Complete the following expressions to make them perfect squares:
   (a) $x^2 - 6x +$ __________ $= (x ---)^2$
   (b) $y^2 + 14y +$ __________ $= (y + -)^2$
   Using (a) and (b) above, determine the centre and radius of a circle whose equation is $x^2 - 6x + y^2 + 14y + 22 = 0$
8. Find the centre and radius of each of the following circles:
   (a) $x^2 + y^2 + 10x + 18y + -38 = 0$.
   (b) $x^2 + y^2 - 16x + 24y + 127 = 0$.
   (c) $x^2 - 15x + y^2 - 7y + 19.5 = 0$.
   (d) $x^2 + y^2 - 26y + 14x = 38$.

Mixed Exercise 3

1. Calculate the probabilities of events listed below:
   (a) Getting two heads with two tosses of a fair coin.
   (b) Drawing a heart from a pack of cards.
   (c) Scoring a sum of more than 8 with two dice.
   (d) Drawing an ace from a pack of cards.
   (e) Drawing a red ball from a bag containing 5 black and 11 red balls.
   (f) Selecting at random a boy from a classroom of 23 boys and 19 girls.
2. Two dice are thrown. Copy and complete the following table.

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	
3, 1	3, 2	3, 3	3, 4		
4, 1	4, 2	4, 3			
5, 1	5, 2				
6, 1					

   (a) What is the probability of the first number being 1?
   (b) What is the probability of the second number being 3?
   (c) What is the probability of getting the same score on both dice?
   (d) What is the probability of the second number being greater than the first?
   (e) What is the probability of scoring a total of 8 or more?
3. The following values p and q, shown in the table below, are thought to obey a law of the form $q = ap^2 + b$.

p	1	2	3	4
q	4.2	8.1	15.4	25.0

   By drawing a suitable line graph, show that this is true and estimate the values of a and b.
4. An alloy consists of three metals A, B and C in the ratio 4 : 4 : 3 respectively by weight.
   (a) Calculate the weight of each metal in 100 kg of the alloy.
   (b) Find expressions for the weight of each metal in $x^2 + 5x - 3$ kg of the alloy.

(c) Out of $x^3 + y^3$ kg of the same alloy, work out expressions for the weights of A, B and C. Hence, deduce the weight of $5A^2 + B^2 - 3C$ kg of the metal in terms of x and y.

5. There are 2 pairs of white socks and one pair of green socks in a box in a dark room. What is the probability that if a person picks two socks from the box without replacement they will be of different colours?

6. A box contains 10 resistors, 2 which are defective. Two resistors are taken out at random with replacement. Find the probability that:
   (a) they are both defective.
   (b) one is defective and the other is not.

7. A contractor makes concrete by mixing cement, sand and gravel in the ratio 2 : 3 : 7. This contractor needs 2 400 kg of concrete to complete a building. If 1 kg of cement costs sh 2.00, how much money will he spend on cement?

8. Use the binomial expansion to evaluate $(5.998)^4$ correct to 6 decimal places.

9. A bag contains 5 red, 4 white and 3 blue beads. Three beads are selected at random without replacement. Find the probability that:
   (a) the first red bead is the third bead picked.
   (b) the beads selected were red, white and blue, in that order.

10. what ratio by volume must a liquid weighing 0.9 kg/litre be mixed with a liquid weighing 0.7 kg/litre in order to make a mixture weighing 0.75 kg/litre?

11. In what ratio must 'Meru' snuff costing sh. 2.50 per 50 g roll be mixed with 'Kamba' snuff costing sh. 1.75 per 50 g roll so that by selling the mixture at sh. 2.50 per 50 gm a profit of 25% is made?

12. Determine the approximate values of a and n if the following values of x and y obey a law of the form $y = ax^n$.

x	2.1	2.6	3.0	3.5	4.2
y	7.6	10.7	14.4	18.5	23.1

13. One letter is chosen at random from each of the words WANGUI, WANJIKU and NYAMBURA. Show that the probability that all the three letters are the same is $\frac{1}{84}$. Find also the probability that just two of the letters are the same.

14. The relationship between x and y is such that $y = a\sqrt{x} + c$, where a and c are constants.

The table below gives corresponding values of x and y.

x	10	20	30	40	50	60
y	2.3	2.7	3.1	3.4	3.6	3.8

By drawing a suitable line graph, determine the values of a and c. From your graph, determine the value of x when y = 2.9.

15. Find the coefficient of x^5 in the expansion of $(2 - 3x)^8$.
16. Obtain the first four terms of the expansion of $(1 + \frac{x}{12})^6$ in ascending powers of x. Use your expansion to evaluate $(\frac{5}{4})^6$.
17. Find the coefficient of x^3y^3 in the expansion of $(2x - 3y)^6$.
18. Expand: (a) $(4x - y)^5$ (b) $(1 + \frac{1}{4}x)^6$
19. Coffee at sh. 40.00 per kg is mixed with coffee at sh. 60.00 per kg in the ratio 3 : 1. In what ratio should this mixture be mixed with coffee at sh. 50.00 per kg to produce a mixture costing sh. 47.00 per kg?
20. A colony of bacteria is decreasing at a rate given by the formula $B = A(1 - 0.02)^t$, where t is time in hours. Initially, there were 20 000 000 bacteria. Express the relation $B = A(1 - 0.02)^t$ in linear form. By plotting log B against to estimate after what time:
    (a) the colony will be reduced by half.
    (b) the colony will be reduced to 7 000 000 bacteria.
    (c) Determine the remaining number of bacteria after:
    (i) 50 hours (ii) 70 hours (iii) 85 hours
21. A card is drawn from an ordinary pack of cards. Find the probability of each of the following events:
    (a) Drawing an ace. (b) Drawing a spade.
    (c) Drawing an ace of spades. (d) Drawing an ace or a spade.
22. The variation of the voltage V and admittance y of an electrical circuit are recorded below.

V	0.35	0.49	0.72	0.98	1.11
y	0.45	0.61	0.89	1.17	1.35

By drawing a suitable linear graph, show that the law connecting V and y is of the form $V = ky^n$ and determine the approximate value of k and n. From the graph determine:

(a) the voltage when the admittance y = 0.75.
(b) the admittance when the voltage V = 1.05.

REVISION EXERCISES

Revision Exercise 1

1. A triangle whose vertices ar A(1, 1), B(2, 1) and C(2, 3) is translated by a vector $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ followed by a reflection in the line y = 0. Find the final image.
2. A figure has a rotational symmetry of order 4 about the point (6, –6). Two of its vertices are (8, –4) and (8, –8). Find the other vertices and draw the figure.
3. Show on a graph the region which satisfies the inequalities $x + y \leq 5$, $x \geq 0$ and $y \geq 0$.
4. If $a = \frac{x + y}{x - y}$ and $b = \frac{1 - a}{1 + a}$, express b in terms of x and y in its simplest form.
5. A square ABCD is rotated through an angle of 60^0 about one of its vertices. Find the angle between the image and the object positions of a diagonal.
6. (a) Show that P (2, 3, 5), Q (3, 4, 6) and R (5, 6, 8) are collinear.

   (b) Given that $\mathbf{a} = \begin{pmatrix} 3 \\ 6 \\ 15 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ r - 4 \\ 20 \end{pmatrix}$ are parallel vectors, find the value of r.
7. Use the matrix method to solve the following simultaneous equations:
   $x + 2y = 4$
   $2x + y = 7$
8. Solve the following quadratic equations:

   (a) $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$ (b) $x^2 - \frac{2}{3}x - \frac{1}{3} = 0$ (c) $6x^2 + 7x + 2 = 0$
9. If Peter gives a quarter of his money to John, John will have twice as much money as Peter. If John gives q shillings to Peter, Peter will have thrice as much as John. Taking the initial amounts owned by Peter and John to be x and y respectively, express:
   (a) y, and,
   (b) q, in terms of x.
10. An open belt passes over two pulleys whose radii are 2.5 cm and 3.5 cm. If their centres are 18 cm apart, calculate the length of the belt.
11. A point Q divides a line PR internally in the ratio2 : 1 and a point T divides the line internally in the 3 : 1. In what ratio does T divide PQ?
12. Without using tables, find the value of $\frac{\log 128 - \log 18}{\log 16 - \log 6}$
13. Given that $\sin A = \frac{4}{5}$, $\cos B = \frac{5}{13}$ and A and B are acute angles, calculate the value of the following without using tables:

(a) $\sin A \cos B + \cos A \sin B$
(b) $\cos A \cos B - \sin A \sin B$
(c) $\tan A \tan B$
(d) $\sin^2 A + \cos^2 A$

14. Give the co-ordinates of the images of the following points when rotated through -90^0 about $(-2, 3)$:
(a) $(-5, 0)$ (b) $(6, 8)$ (c) $(4, -2)$ (d) $(-4, -4)$

Revision Exercise 2

1. Draw a circle of radius 4 cm and mark a point 7 cm from the centre. Draw the tangents from the point to the circle and measure their lengths.

2. Solve the equations:
(a) $x^3 - 2x^2 - 3x = 0$ (b) $6x^2 - x - 2 = 0$

3. One man can do a piece of work in $4\frac{1}{2}$ hours and another man can do it in 6 hours. How long would they take if they worked together?

4. Solve the simultaneous equations;
$3x - y = 5$
$5x - 2y = 9$

5. Two points A and B are such that $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$. Give the distance AB.

6. **H** is a half-turn about (0, 0) and **E** an enlargement scale factor 3 centre origin. Find the image of a triangle whose vertices are P(1, 1), Q(2, 1) and R(1.5, 2) under the successive transfornations:
(a) **EH**
(b) **HE**
(c) What is the area of the image of this triangle under **EEH**?

7. Show on a graph the region which satisfies the inequalities;
$2x + y \geq 3, x + y \leq 2, y \geq -2.$

8. If $x = \dfrac{2a - 3}{3a + 2}$ and $a = \dfrac{3y - 2}{3y + 3}$, express x in terms of y.

9. The volume V of a cylinder of base radius r and height h varies jointly as h and r^2. If $V = 75.36 \text{ cm}^3$ when h = 6 cm and r = 2 cm, find r when $V = 226.08 \text{ cm}^3$ and h = 8 cm.

10. Find the limits within which the areas and the perimeters of the following must lie if their sides are measured to the nearest cm:
(a) a square of side 10 cm.
(b) an equilateral triangle of side 10 cm.

11. Two translations $\mathbf{T}_1$ and $\mathbf{T}_2$ are represented by the vectors $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

respectively. If L is the point (4, 5), find $\mathbf{T}_1$(L) and $\mathbf{T}_2$(L).
It is further given that $T_3 = T_1T_2$ find:
(a) $\mathbf{T}_3$(L)
(b) $\mathbf{T}_1\mathbf{T}_3$(L)
(c) $\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$(L)

12. A prism has a regular decagonal cross-section.
 (a) How many axes of symmetry has the solid?
 (b) What is the order of its rotational symmetry?
 (c) How many planes of symmetry has the prism?

13. A point Q divides a line PR in the ratio 2 : –1 and a point T divides the line QR in the ratio 3 : –1. In what ratio does T divide PQ?

14. (a) Draw the graphs of $y = x^2 + 4x + 1$ and $y = x^3 - 3x + 1$ on the same axes for a suitable range of values of x. Use your graph to determine the three distinct roots for which $x^3 - 3x + 1 = x^2 + 4x + 1$.
  (i) Does the graph of $y = x^2 + 4x + 1$ have a minimum or maximum value?
  (ii) Give the equations of those tangents to the curves which are parallel to the x-axis.
 (b) Determine the points of intersection between the curves and the lines:
  (i) $y = 10$, and,
  (ii) $y = x$
 (c) Give the equation of the line of symmetry of the curve $y = x^2 + 4x + 1$.

15. A fair coin and a die are thrown once. Find the probability of getting:
 (a) a head and a 5.
 (b) a tail and a number greater than 4.
 (c) a head and an even number.

Revision Exercise 3

1. (a) If $v^2 = \sqrt{\frac{1+c^2}{r^2}} + \frac{r}{3}$, make c the subject of the formula.
 (b) Make r the subject of the formula;
 $n = \frac{1}{2x}\sqrt{\frac{r}{k}}$

2. The amount of rainfall to the nearest mm recorded in a week in a school weather station was 11 mm, 10 mm, 6 mm, 9 mm, 15 mm, 12 mm, 10 mm. Find the maximum and the minimum possible values of the mean amount of rainfall for the week.

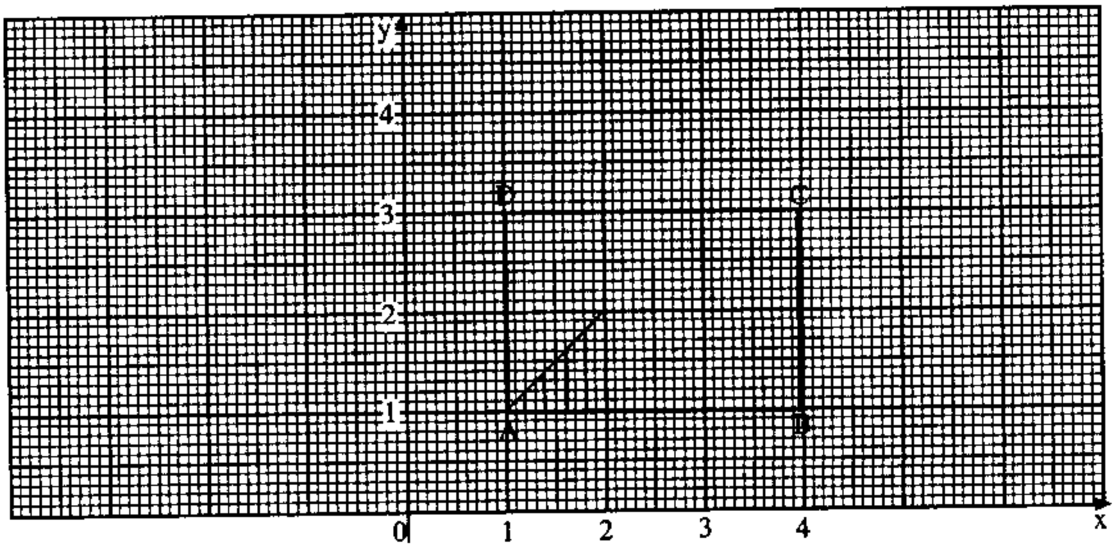
3. P, Q and R are points on a level ground such that PQ = 0.55 km, QR = 0.64 km and ∠PQR = 120^0. Calculate the distance from P to R.

4. A rent-a-vision company rents video sets for sh. 100 per week, or part thereof. The company buys a set for sh. 20 000 and it costs sh. 5 per week to maintain. After how many weeks will the company recover the cost of buying and maintaining a set?
5. Given that the co-ordinates of two points A and B are (6, 5) and (–4, –8) respectively, find the co-ordinates of a point P which divides AB internally in the ratio 2 : 5.
6. The area A of a sector of a circle of radius r varies jointly as r^2 and θ, the angle of the sector at the centre of the circle. If $A = 30$ cm^2, $r = 8$ cm and $\theta = 24^0$, find A when $\theta = 48^\circ$ and $r = 4$ cm.
7. Solve the equation.

   $\frac{x-1}{1} = \frac{1}{2x-3}$
8. Solve the simultaneous equations:

   $y^2 = x - 1$, and,

   $y^4 + 3x = 7$ for real values of x and y.
9. If $\log(4x - 9) = 3 \log 3$, find x.
10. A bag contains 3 black balls and 6 white ones. If two balls are drawn from the bag one at a time, find:
    (a) the probability of drawing a black ball and a white ball:
        (i) without replacement.
        (ii) with replacement.
    (b) drawing two white balls:
        (i) without replacement.
        (ii) with replacement.
11. Three men working 8 hours daily can complete a piece of work in 5 days. Find how long it will take 10 men working 6 hours a day to complete the same work.
12. The shape in the diagram below is enlarged by a scale factor –2 and $\frac{1}{2}$ about the origin. A point is selected at random in the two images so formed. Find in each case the probability that the point lies in the triangle.



13. Two circles of radii x and y have their centres C units apart. C being greater than x + y. Calculate the length of:
    (a) the exterior common tangent.
    (b) the interior common tangent.
14. In a building the weight which can be supported by a cylindrical iron column varies directly as the fourth power of the diameter of the iron and inversely as the square of its length.
    (a) If the length is halved, in what ratio must the diameter be changed to carry the same weight?
    (b) If the diameter is doubled and the weight to be carried trebled, find the necessary percentage change in the length of the column.
15. Use the graphical method to solve the following equations:
    (a) $2 \sin 2\theta + 1 = 0, \; 0 \leq \theta \leq 360^0$
    (b) $\sin \theta - \cos \theta = 0, \; 0 \leq \theta \leq 360^0$
    (c) $\cos 2\theta - 2\sin 2\theta = 0, \; 0 \leq \theta \leq 360^0$
16. The amount of money A invested over a period of n years at r% compound interest is given by the formula $A = P(1 + r)^n$.
    (a) By taking logarithms to both sides of the expression, obtain a linear expression in A, P, n and r.
    (b) If P = sh. 20 000, r = 15% and n = 10, use the expression you obtained in (a) to draw a straight line graph.
    From your graph, deduce the amount of money due at the end of:

    (i) 5 years. (ii) $8\frac{1}{2}$ years.

    (iii) $9\frac{1}{2}$ years. (iv) $11\frac{1}{2}$ years. (v) 2 years.

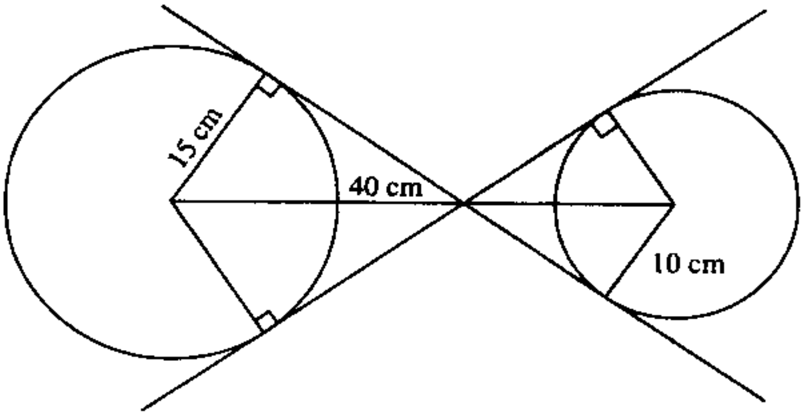
Revision Exercise 4

1. Three transformations, **R**, **M** and **T** are defined as;
   R : A rotation of $+90^0$ about the origin.
   M : A reflection in the line y = x.
   T : A translation represented by $\begin{pmatrix}-5\\2\end{pmatrix}$

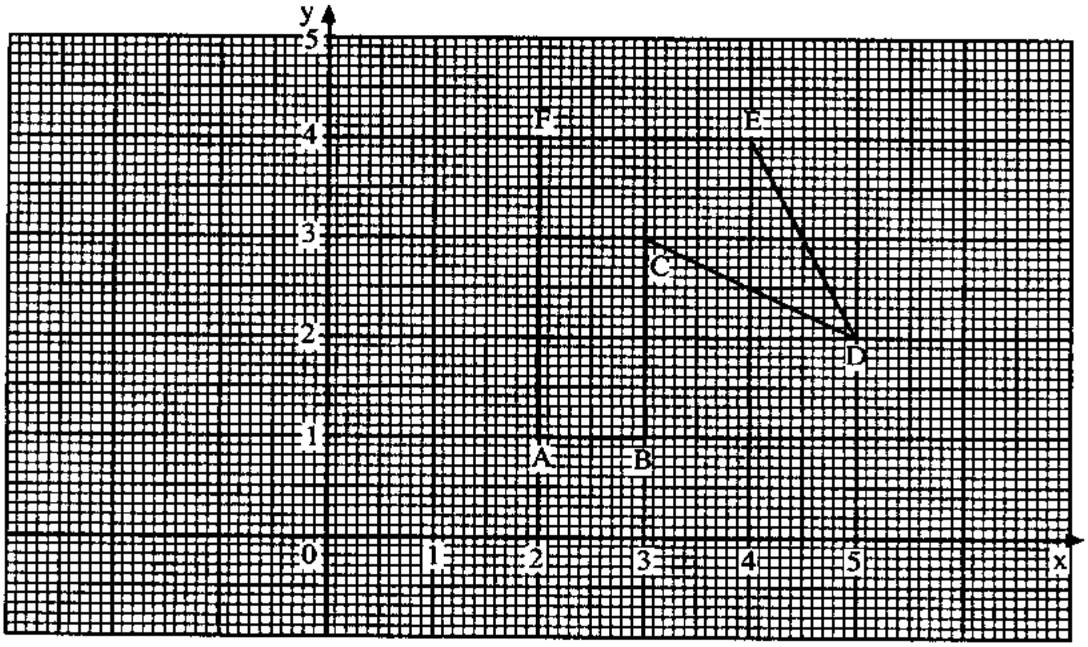
   Find the images of P(2, 3), Q(–3, 3) and S(0, 5) under the transformation **TMR**.
2. Find the co-ordinates of the vertices of the image of a kite whose vertices are; P(0, 8), Q(3, 3), R(0, 1) and S(–3, 3) when rotated about the origin through:
   (a) -90^0
   (b) 180^0
3. The points P(–2, 5) maps onto P′(1, 9) under a translation $\mathbf{T}_1$. If P′ is mapped onto P″ under a translation T_2 given by $\begin{pmatrix}-4\\-1\end{pmatrix}$, find:
   (a) the co-ordinates of P″.
   (b) $\mathbf{T}_3$ given $\mathbf{T}_3$(P) = P″.
4. Using table 6.1, calculate the amount of P.A.Y.E for:
   (a) a person earning sh. 12 890 per month.
   (b) a person earning sh. 30 000 per month.
   (c) a woman earning sh. 24 000 per month and housed by employee.
   (d) a businessman earning sh. 20 000 per month.
5. The volume of a fixed mass of gas is inversely proportional to its pressure. When the volume is 400 cm^3, the pressure is 60 cm mercury. Find:
   (a) the pressure when the volume is 300 cm^3.
   (b) the volume when the pressure is 80 cm.
6. Solve the equation $\cos\theta = 0.6$ for θ between 0^0 and 360^0.
7. Draw the graph $y = 2x^2 + x + 2$. Using the same axes, draw its reflection in the x-axis. What is the equation of the image?
8. Draw the graph of $y = \sin\theta$ for values of θ between 0^0 and 360^0.Use the graph to solve the following equations:
   (a) $5 \sin\theta = 2$ (b) $4 \sin\theta + 2 = 1$
9. The width of a rectangle is one centimetre less than its length. If its area is 20 cm^2, calculate its perimeter.
10. If A and B are (3, 5) and (–4, –5.5) respectively, find the co-ordinates of a point P which divides AB in the ratio 2 : 5.
11. Three variables A, x and t are known to be connected by the relationship $A = A_0 (1 - x)^t$, where A is the amount of substance remaining, A_0 the initial amont of the substance, x the rate of decrease and t the time in

hours. Find the rate that would reduce the substance by half in 5 hours. If the rate is known to be 3%, in how many hours would A be reduced to $\frac{1}{4}$ of its orginal value?

12. The figure below shows a transverse pulley system. The wheels have diameters of 20 cm and 30 cm respectively. If the distance between the centres is 40 cm, calculate the length of the belt.



13. A regular eight-sided spinner has 2 blue sides, 3 red sides and 3 black sides. The spinner is spun and the colour of the side on which it falls noted:
    (a) What is the probability of spinning:
        (i) a red. (ii) a black. (iii) a blue.
    (b) What is the probability of spinning:
        (i) blue then red (ii) blue, then red, then black.
14. The figure below shows an irregular hexagon ABCDEF in the x-y plane. Three transformations **T**, **Q** and **M** are applied to it.



T is a translation given by the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, **Q** a positive quarter turn about the origin and **M** a reflection in the x-axis. Find:

(a) the gradient of the image of AC after T.

(b) the equations of the images of CD, ED after the transformations **MT**, **QMT** and **TMQ**.

(c) the inequalities satisfied by the region enclosed by the images of the lines AB, BC and AC after the transformation **MQ**.

(d) a single transformation equivalent to **MQ**.

15. Triangle ABC is isosceles with AB = AC = 10 cm and ∠BAC = 36°. The triangle is reflected on the edge BC. Find the angle between the new position of AB and the initial position of triangle AB.

16. A triangle has vertices at P(–1, 3) Q(–3, 5) and R(–2, 8). The triangle is reflected in the line y = 3x – 1 to get triangle P′Q′R′. It is then translated by a vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ to get P″Q″R″. Draw triangles PQR, P′Q′R′ and P″Q″R″.

Revision Exercise 5

1. Make u the subject of the formula; $x = \frac{uv}{u + 2w}$

2. The principal varies directly as the interest and inversely as both rate and time. If principal is sh. 120, interest sh. 24, rate 10% and time 2 years, find the principal in terms of interest, rate and time.

3. Solve the triangle ABC in which AB = 6 cm, BC = 9 cm and angle ABC is 35°. Find the area of the triangle.

4. What is the probability of throwing a total of 12 in a single throw of two dice?

5. Find the sum of the first 10 terms of the arithmetic series;

   $1 + \frac{1}{2} + 0 - \frac{1}{2} + \ldots$

6. If $\mathbf{U} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$, $\mathbf{V} = \begin{pmatrix} -3 & 1 & 5 \\ 0 & 2 & -4 \end{pmatrix}$ and $\mathbf{W} = \begin{pmatrix} 2 & 4 \\ 1 & 5 \\ 3 & -1 \end{pmatrix}$, find where possible

   (a) **UV** (b) **VU** (c) **VW** (d) **WV**

7. What is the probability of getting a sum greater than 10 when two fair dice are thrown once?

8. Find the inverse of: (a) $\begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -p & -q \\ 2 & 3 \end{pmatrix}$

9. Use the graph of $y = x^2 + 4x + 3$ to solve:

   (a) $x^2 + 4x + 3 = 7$ (b) $x^2 + 4x - 1 = 0$

10. In the expansion of each of the following in ascending powers of x, find the term indicated:

    (a) $(x - y)^7$, 5th term.
    (b) $(2x - 3y)^8$, 4th term.
    (c) $(2x - \frac{1}{3}y)^9$, 5th term.
    (d) $(2 - \frac{1}{3}y)^9$, 3rd term.
    (e) $(1 - \frac{1}{5}x)^6$, 3rd term.

11. The perimeter of a rectangle is given as 12 cm to two significant figures. The length is 3.4 cm to two significant figures. What is the least and the greatest value of the breadth?

12. The resistance of an electrical conductor is partly constant and partly varies as the temperature. When the temperature is 20°C, the resistance is 55 ohms. When the temperature is 28°C, the resistance is 58 ohms. Find:

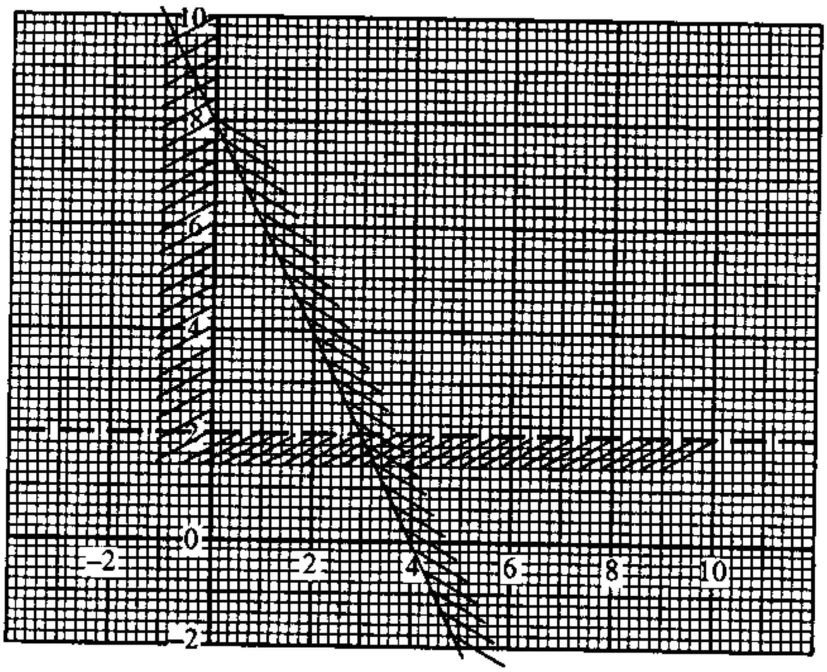
    (a) the resistance when the temperature is 60°C.
    (b) the temperature at which the resistance would be 60 ohms.

13. Two gear-wheels are to be fixed at a distance of 60 cm from each other. If the diameters of the gears are 40 cm and 20 cm, calculate the length of the drive chain required to connect them.

14. M_1 and M_2 are reflection in the line $y = x + 1$ and $x = 3$ respectively. Triangle P has vertices A(2, 1) B(4, 3) and C(–1, 1). State the co-ordinates of the image of P under:

    (a) M_1M_2
    (b) M_2M_1

15. State the inequalities satisfying the unshaded region shown in figure 1 below.



16. During an epidemic, the rate at which a virus was spreading was found to be given by the relation $P = Q(1 + 0.05)^t$, where Q was the initial number of people infected, P the current number of patients and t the time in hours. At 12.00 noon when the epidemic was first detected already, people were already infected. When the medical team arrived three hours later, the number of patients had increased to 116. Draw a suitable linear graph and use it to estimate:
    (a) the number of people infected by midnight on the day preceding the discovery of the epidemic.
    (b) the time the number of patients had grown to 60.
    (c) the duration the infection had been going on by 1500.

Revision Exercise 6

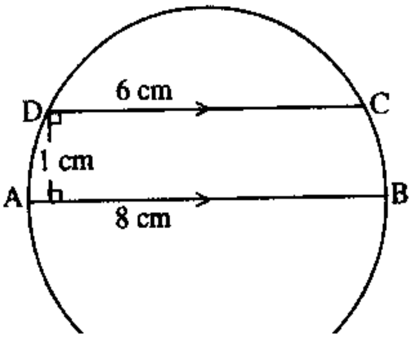
1. The area of a square plot of land is stated at 225.0 m^2. Within what limits does the length of its side lie?
2. Use binomial expansion to find the first four terms of each of the following:
   (a) $(a + b)^8$ (b) $(a - b)^7$ (c) $(a + 2b)^6$
3. A straight road up a hill side makes an angle of 15^0 with the horizontal. How far must a man walk along the road in order to rise a vertical height of 50 m?
4. The probability of Halima winning a game is $\frac{3}{8}$. She plays four games in succession. Find the probability of her winning three games.
5. If $A'(-6, 4)$, $B'(-8, 4)$, $C'(-10, 2)$ and $D'(-8, 2)$ are vertices of the image of a parallelogram with vertices at A(3, –2), B(4, –2), C(5, – 1) and D(4, –1), describe:
   (a) the transformation that maps the parallelogram ABCD onto $A'B'C'D'$.
   (b) the transformation that would map $A'B'C'D'$ onto ABCD.
6. Draw the graph of $y = 3x^2 + 2x - 5$ for values of x from x = –5 to x = 5
   Use your graph to solve:
   (a) $3x^2 + 2x - 1 = 0$ (b) $3x^2 + 4x - 5 = 0$
   (c) $3x^2 - 5 = 0$ (d) $3x^2 + 6x = 0$
7. Four points, A, B, C and D in space are such that **AB** = 6**i** + 10**k**, **BC** = –6**i** + 8**j** + 10**k** and **AB** = 2**CD**. How far apart are points B and D?
8. In a game of darts, a player is required to hit a maximum of triple scores, with each of the three darts, of the numbers 1 to 20. Find the maximum score a player can get.
9. The area of a rectangular *shamba* is 10.2 ha to the nearest 0.1 ha. If the length of the *shamba* is 180 m to the nearest 10 m, calculate the least and the greatest possible values of the width.

10. Solve the following simultaneous linear equations:
    (a) $3p - 4q = 17$
        $p + q = -1$
    (b) $p + q + r = 0$
        $2p + q - r = -1$
        $p + 2q + r = -2$
11. The kinetic energy, E joules, of a body of mass m varies directly as the mass m and as the square of its velocity v ms^{-1}. For a body of mass 0.1 kg, $E = 0.2$ when $v = 2$. Find E when $v = 4$.
12. In an experiment, the values of two quantities p and q were observed and the results recorded as below:

p	0	2	4	6	8	10
q	0.49	0.30	0.24	0.20	0.16	0.13

It is known that q is related to p by an equation of the form $q = \frac{a}{b+p}$, where a and b are constants. By drawing a suitable straight line graph estimate the values of a and b.

13. If S′ is the image of the point S under a reflection in the line *l*, find $|SS'|$ when:
    (a) S is the point (4, 2), *l* is $x = 5$
    (b) S is the point (6, 2), *l* is $x = 4$
    (c) S is the point (–2, 4), *l* is $y = -x$
    (d) S is the point (5, 2), *l* is $x + y = 8$
14. A quantity P varies partly as t and partly as the square of t. When $t = 20$, $P = 45$ and when $t = 24$, $P = 60$.
    (a) Express P in terms of t.
    (b) Find P when $t = 32$ and t when $P = 75$.
15. The figure below shows two parallel chords AB and DC. AB = 8 cm and DC = 6 cm. If the chords are 1 cm apart, calculate the radius of the circle.



Revision Exercise 7

1. Find the length of the diagonal of a cube whose length is 10 cm.

2. Determine the inverse of the matrix $\begin{pmatrix} 6 & 4 \\ 10 & 8 \end{pmatrix}$.

3. What term should be added to the following expressions so as to make them perfect squares? Of what expression is each sum the square?

   (a) $x^2 + 8x$ (b) $c^2 - 11c$ (c) $16x^2 + 25y^2$

   (d) $p^4 + 16$ (e) $1 - 16x^2$

4. A radiocassette is sold under hire purchase terms. A deposit of sh. 560 is required and a monthly instalments of sh. 400 for one year. If the cash price of the same radio is sh. 3 820, find the difference between the cash and hire purchase prices.

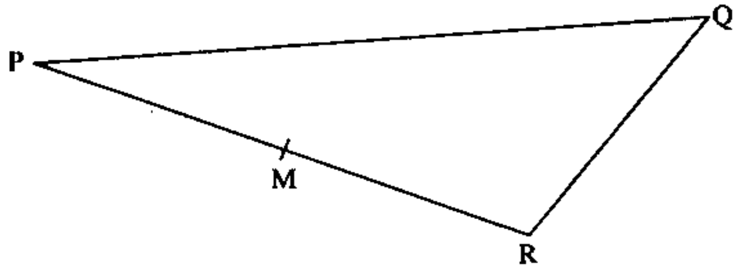
5. Use matrix method to solve the pairs of simultaneous equations:

   (a) $3x - 4y = 10$ (b) $0.1p + 2.5q = 6$

   $x + 3y = -1$ $2.3p - 8q = 7$

6. A substance is made up of three chemicals A, B and C in the ratio 5 : 4 : 1. A chemist wants to make 500 g of the substance. He can only get chemical A by extracting it from a compound which contains 20% of it. How many grams of the compound does he need?

7. Triangle PQR is given a half-turn about M, the midpoint of PR.



   Show that:

   (a) PQRQ′ is a parallelogram, where Q′ is the image of Q.

   (b) the diagonals of the parallelogram bisect each other.

8. Three cards are picked at random without replacement from an ordinary pack of cards. Find the probability that the cards are Jacks, Kings and Queens:

   (a) in that order.

   (b) in any order.

9. Find the lower and upper limits of the differences between:

   (a) 13.1 cm and 18.7 cm.

   (b) 0.88 km and 0.093 km.

   (c) 2.53 g and 4.22 g.

   (d) 8.3 m and 15.9 m.

10. In a batch of 18 components, 4 are known to be defective. If random selection is made of 3 components (one at a time) what is the probability that all three will be defective?
11. (a) If p varies directly as r and inversely as the square root of q, find the percentage change in p if r increases by 40% and q decreases by 36%.
    (b) If p = 8 when r = 6 and q = 9, find p in terms of r and q.
12. From a point A due west of a vertical post, the angle of elevation of the top of the post is 60^0. From a point B due south of the post, the angle of elevation of the top of the post is 60^0. The points A, B and the foot of the post are in the same horizontal plane. If AB = 6 m and A and B are equidistant from the foot of the post, find the height of the post to 1 d.p.
13. If a weight is attached to a string and swung in a horizontal circle, the tension, T newtons, in the string varies directly as the square of the speed vms^{-1}, and inversely as radius r metres, of the circle. If the radius is doubled and the speed halved, find the percentage change in the tension.
14. The data given below were obtained from an experiment on torsion.

T (NM)	845	895	945	995	1 045	1 095
θ (deg)	12.0	14.2	16.8	20.0	27.1	27.1

It is believed that the law connecting the variables T and θ is of the form $T = k\theta^n$ where k and n are constants. By drawing a suitable graph, estimate:

(a) the values of k and n.
(b) the values of T when $\theta = 15.5^0$.
(c) the value of θ when T = 1 000 Nm.

Revision Exercise 8

1. A variable p is partly constant and partly varies as another variable q. When q = 3, p = 1 and when q = 4, p = 10. Find p in terms of q. Hence, find p when q = 6.
2. The diameter and height of a cylinder are stated as 21.7 cm and 15.6 cm respectively. Find the limits within which its volume lies.
3. Two cards are drawn at random from a pack. Find the probability that they are:
   (a) both kings.
   (b) both spades.
   (c) both of the same unit.
4. Calculate the annual income of a woman who has to pay sh. 4 280 as P.A.Y.E. every month (use the table of tax rates in chapter 6).

5. Evaluate:

   (a) $(1 \quad 2)\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $(3 \quad 1 \quad 2)\begin{pmatrix} 2 & 1 \\ 4 & 0 \\ 2 & 2 \end{pmatrix}$

6. Machine A can complete a piece of work in 6 hours while machine B can complete the same work in 10 hours.
   (a) If the two machines are working together, how long will it take to complete the work?
   (b) If both machines start working together and machine A breaks down after 2 hours, how long will it take machine B to complete the rest of the work?

7. The following values of x and y are suspected to obey the law $x = ay^n$.

y	5	10	15	20	25
x	65	182	335	519	724

   By plotting a suitable line graph, estimate the values of a and n.

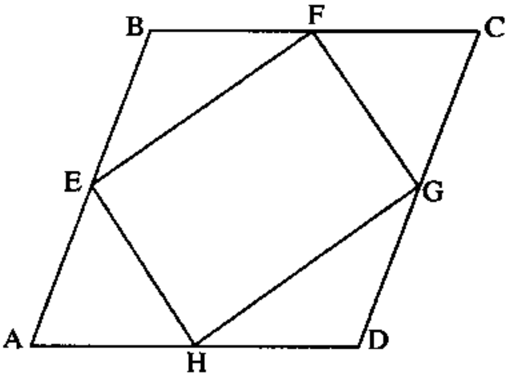
8. Find the length of a transverse common tangent to two circles of radii 8 cm and 4 cm whose centres are 14 cm apart.

9. Find the equation of the image of the line whose gradient is 3 and passes through the point (2, 1) under:
   (a) translation by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
   (b) reflection in the line through the points (3, 2) and (1, 4).
   (c) rotation through -90^0 about (4, 2).

10. A liquid 95% pure alcohol and the rest water. In what ratio must it be mixed with water in order that the mixture contains 70% pure alcohol?

11. Find the value of x in $(\log x)^2 - \log x^{15} + 56 = 0$

12.



In the figure above, E, F, G, H are midpoints of the sides of a quadrilateral ABCD. Show that EFGH is a parallelogram.

13. Solve the following pairs of simultaneous equations and interpret your results geometrically:

    (a) $3x - 1\frac{1}{2}y = 3$ (b) $3x - 1\frac{1}{2}y = 3$

    $x - \frac{1}{2}y = 1$ $x - \frac{1}{2}y = 2$

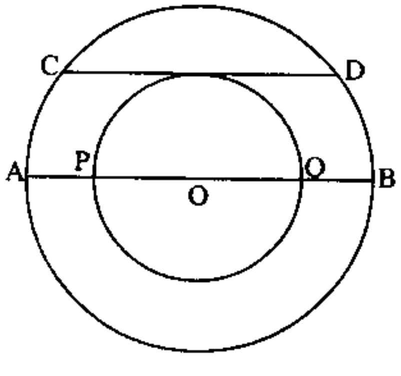
14. If one root of the equation $12x^2 + 9x + B = 0$ is $\frac{3}{4}$, find B.

    Hence, find the other root.

15. Three equal unbiased dice are tossed simultaneously. Calculate the probabilities that:
    (a) two fives and one six will appear.
    (b) a total of 15 will be thrown.

Revision Exercise 9

1. Find the distance between the points 3**j** + **k** and 2**i** + **j** + **k**.
2. Three points P, Q and R lie on a circle. The tangent at Q meets RP produced at S. Show that ∠RQS = ∠SPQ.
3. The masses of two objects to the nearest kilogram are 51 kg and 43 kg. Find the limits within which the sum of their masses lies.
4. One pipe can empty a cistern in 8 hours, another in 6 hours and a third in 3 hours. How long will it take to empty the cistern if all the three pipes are used simultaneously?
5. The figure below shows two concentric circles with diameters AB = 10 cm and PQ = 5 cm respectively. Find the length of CD which is a tangent to the inner circle and also a chord of the larger circle.



6. A company borrowed sh. 500 000. The interest on the loan was $1\frac{1}{2}\%$ per month charged on the outstanding balance. The company repaid the loan in monthly instalments of sh. 40 000 each. If this amount included both the interest on the loan and part payment of the principal sum borrowed, calculate the amount of money which the company still owed after 6 months.

7. A brand X of tobacco contains 4.0% nicotine. Another brand Y contains 12% nicotine. In what ratio should the two brands be mixed to produce a brand with 6% nicotine?

8. A variable t is partly constant and partly varies inversely as the square of s. When s = 2, t = 0 and when s = 1, t = 3. Find t when s = $\frac{1}{2}$.

9. Use the first three terms of a binomial expansion to find the approximate values of:
   (a) 1.01^6 (b) 1.005^4 (c) 1.98^8

10. The power generated by a machine is given by $P = kN + \frac{a}{N}$, where k and a are constants. Find the values of k and a if P = 11 when N = 2 and P = 9 when N = 1.5.

11. Find an expression for b in terms of c if one root of the equation $x^2 + bx + c = 0$ is three times the other.

12. Find the rotation which maps the right-angled triangle with vertices L(–4, –5), M(–6, –5) and N(–4, –3) onto the triangle with vertices L′(–6, –3), M′(–2, –5) and N′(–4, –3). Find the image of the point (3, –4) under the same rotation.

13. The table below gives corresponding values of P and V which are suspected to obey the law of the form $P = aV^2 + bV$, where a and b are constants.

V	0.52	2.58	5.25	8.0	9.5
P	4.6	38.5	121.3	235.1	324.2

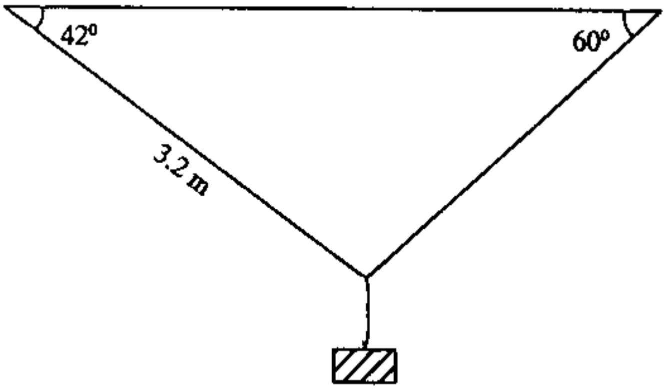
By plotting a suitable linear graph, show that the law is satisfied, and find the values of a and b. Hence, find:
   (a) the value of P when V = 6.8.
   (b) the value of V when P = 185.

14. For each of the following sequences, state whether it is arithmetic, geometric or neither:
   (a) $a, a^2, a^4, a^6, \ldots$
   (b) $5, \frac{5}{7}, \frac{5}{49}, \frac{5}{343}, \ldots$
   (c) $x, x + 1, x + 2, x + 3, \ldots$
   (d) $3, 3 \times 2, 3 \times 3, 3 \times 4, \ldots$
   (e) $a, (a + x)^2, (a + x)^3, (a + x)^4, \ldots$

Revision Exercise 10

1. Rewrite the expression $(p + 2 - r)(r + p - 2)$ in the form $A^2 - B^2$.

2. Given that **a** = 2**i** – **j** + **k** and **b** = 3**i** + 5**j** – **k**, find:
   (a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) $|\mathbf{a}+\mathbf{b}|$ (d) $|3\mathbf{a}+2\mathbf{b}|$ (e) $|3\mathbf{a}|+|2\mathbf{b}|$
3. In a class John and James were told that their masses were 52 kg and 53 kg respectively. James says that he is 1 kg heavier than John. Comment on the accuracy of James' statement.
4. Give the co-ordinates of the images of the following points when rotated through 180^0 about (4, 5):
   (a) (–4, 0) (b) (5, 7) (c) (2, –1) (d) (–2, –2)
5. Construct the pair of tangents from a point which is 6 cm from the centre of a circle radius 4 cm. Measure the length of the tangent.
6. The cost of printing a book is partly constant and partly varies as the number of pages. If a book has 200 pages, the cost is sh. 400 and if it has 100 pages, the cost is sh. 240. Find the cost of printing a book with 400 pages.
7. The dimension of a rectangle are given as 6.2 m by 2.5 m. Find the minimum possible values for the area of the rectangle.
8. A stone is hung from a horizontal beam by two strings. The longer string makes an angle of 42° with the horizontal and is 3.2 m long. If the shorter string makes an angle of 60° with the horizontal, calculate its length (see the figure below).



9. A tea blender has two sorts of tea, the Housewife choice and the family special. One kilogram of Housewife choice and one kilogram of Family special cost sh. 70 and sh. 64 respectively. In what ratio should he mix the two to make a blend which costs sh. 68 a kilogram?
10. Simplify each of the following:
    (a) $\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$
    (b) $\dfrac{1}{1+\sqrt{2}}+\dfrac{3}{1-\sqrt{2}}$

(c) $3\sqrt{2}(\sqrt{2} - \sqrt{3})$

(d) $(1 - \sqrt{2})(1 - \sqrt{3})$

11. A schoolboy reckons that the value of his bicycle depreciates by 10% of its value at the beginning of each year. The bicycle cost sh. 3 500 when new. Find the value of the bicycle after 4 years.

12. Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ -4 & 4 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$, find the matrix product **AB**.

13. The square whose vertices are P(3, 2), Q(5, 2), R(5, 4) and S(3, 4) is rotated through $+90^0$ to give its image whose vertices are $P'(2, 5)$, $Q'(2, 7)$ and $S'(0, 5)$.

(a) Find the centre of rotation.

(b) If the square $P'Q'R'S'$ is reflected in the line $y = x + 4$ to give square $P''Q''R''S''$, find the co-ordinates of P'', Q'', R'' and S''.

14. The first term of an arithmetic sequence is 2. The first term of a geometric sequence is also 2 and its common ratio equals the common difference of the arithmetic sequence. The third term of the geometric sequence exceeds the square of the first term of the arithmetic sequence by 124. Find:

(a) the common difference.

(b) the sum of the first 10 terms of the arithmetic sequence.

15. The internal radius of a pipe with a uniform cross-section is 2.8 cm. Water is flowing out of it at the rate of 3 cm per second. Find the amount of water discharged by the pipe in 7 minutes. Give your answer in litres.

16. In an experiment on moments, a bar was loaded with a mass m at a distance s cm from the fulcrum. The results of the experiment were as shown in the table below:

s(cm)	25	27	29	31	33
m(kg)	21.5	19.6	18.4	17.2	16.6

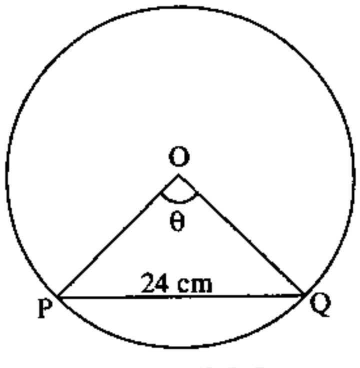
Verify that a law of the form $m = as^n$ is obeyed, where a and n are constants. Find the values of a and n.

Revision Exercise 11

1. If $p^2 + 10p + 30$ is equivalent to $(p + a)^2 + b$, determine the values of a and b.

2. The number of revolutions R a wheel of a car makes is directly proportional to the speed S and inversely proportional to its diameter d. If d = 60 cm when R = 150 and S = 20 kmh^{-1}, find R when d = 36 cm and S = 35 kmh^{-1}.

3. The area of a rectangle is 27 cm^2 to the nearest whole number. What is the possible length if the width is 3 cm to the nearest whole number?
4. A sewing machine valued at sh. 25 000 can be bought by cash at a discount of 10% or by instalments whereby a deposit of sh. 3 000 is paid followed by 15 monthly instalments of sh. 1 500 each. Find:
   (a) the cash price of the machine.
   (b) the hire purchase price of the machine.
5. A petrol pump bought for sh. 12 500 depreciates each year at a rate of 5% of its value at the beginning of the year. What will be its value after five years?
6. Find the co-ordinates of the points at which the straight line $x - y = 5$ intersects the curve $xy = 24$.
7. The variables s, x and w are connected by the expression;
   $x = \frac{1}{888}(s - \frac{3}{5}w)w^2$.
   (a) Make s the subject of the formula.
   (b) If $s = 25$ w, express s in terms of x.
8. If $\mathbf{L} = \begin{pmatrix} 9 & -4 \\ 1 & 7 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 5 & 3 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 2 & -4 & 2 \\ -6 & 7 & 1 \end{pmatrix}$ and $\mathbf{K} = \begin{pmatrix} 4 & 2 & 1 \\ -3 & 4 & -1 \\ 1 & 7 & 3 \end{pmatrix}$, find the values of:
   (a) $\mathbf{L}^2$ (b) **LM** (c) **MK** (d) **NK**
9. The resistance to the motion of a moving vehicle is partly constant and partly varies as the square of the speed. At 60 kmh^{-1}, the resistance is 630 newtons and at 40 kmh^{-1}, the resistance is 450 newtons. What will be the resistance at 90 kmh^{-1}?
10. Two dice are thrown, one white and one blue. What is the probability that the number on the white die is greater that than on the blue die?
11. The ratio between the tenth and the seventh terms of a geometric progression is 64. Find the common ratio of the progression.
12. In the following figure, O is the centre of the circle, $\theta = 90^0$ and PQ = 24 cm:



Find: (a) the area of the sector POQ.
(b) the area of the minor segment PQ.

13. A boarding school uses 15 bags of maize, 8 bags of beans, 16 bags of maize flour and 4 bags of rice in the first term. The prices are sh. 1 200, sh. 2 400, sh. 1 400 and sh. 1 400 respectively. In the second term, the school uses 16 bags of maize, 10 bags of beans, 18 bags of maize flour and 5 bags of rice at sh. 1 400, sh. 2 600, sh. 1 600 and sh. 1 500 respectively. In the third term, the school uses 12 bags of maize, 5 bags of beans, 12 bags of maize flour and 3 bags of rice at sh. 1 800, sh. 2 200, sh. 2 000 and sh. 1 500 respectively. Using a matrix method, find the total cost of the foodstuff that year.
14. The population of a town is 100 000. How many years will it take to double if the growth rate is 30 per thousand per year?
15. A man drives for 100 km at 50 kmh^{-1}, then 75 km at 60 kmh^{-1} and a further 80 km at 40 kmh^{-1}. What is his average speed for the whole journey correct to 1 decimal place?
16. The table below gives corresponding values of the volume V and pressure P in a compressed mixture of gases in a cylinder.

V	0.75	1.45	2.95	5.95	8.95
P	198	88	34	12	7

It is believed that these results obey the law of the form $PV^m = c$ where m and c are constants. By drawing a suitable linear graph, verify the law. Use your graph to estimate the values of m and c and hence find the actual law.

Revision Exercise 12

1. With reference to a point O as origin $\mathbf{OP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{OQ} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$. Find **PQ** and determine its magnitude.
2. The radius of a sphere is measured as 6.3 cm. to the nearest 0.1 cm. Calculate the percentage error in the calculation of the volume using this radius.
3. T is a translation which maps point (4, 1) onto (–2, 6). **R** is a rotation of 180^0 about (–1, 2). P is the point (2, 2), Q is (2, –4) and S is (–7, –1). Find the co-ordinates of **T**(P), **R**(Q) and **TR**(Q).
4. The difference between the eighth term and the fourth term of an A.P. is 24. If the first term of this series exceeds the common difference by 4, find the tenth term of the series.

5. The perimeter of a right-angled triangle is 36 cm and the hypotenuse is 15 cm. Find the lengths of the other two sides. Hence calculate the area of the triangle.
6. The sine of an acute angle A is $\frac{2}{\sqrt{13}}$. Without using tables, calculate tan A and cos(180 – A).
7. A and B are two points 5 km apart. C is another point such that $\angle CAB = 68^0$ and $\angle CBA = 80^0$. Find the length CA, CB and the perpendicular distance of C from the line AB.
8. The vertices of a square are P(–3, –1), Q(–1, –1), R(–3, –3) and S(–1, –3). If **M** denotes a reflection in the line y = –x and **T** denotes the translation$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$, find the co-ordinates of the vertices of the image of the square under the composite transformation:
   (a) **MT** (b) **TM**
9. Tap T_1 fills a tank in 8 hours and tap T_2 empties it in 12 hours. How long does it take to fill the tank if both taps are left running?
10. A man earned £ 6 500 in one year. He paid income tax at the rate of sh. 2 in the pound for the first £ 5 808 and sh. 3 in the pound for the rest of his income. Calculate how much P.A.Y.E. he paid.
11. Two toy cars are moving from point A to point B, a distance of 0.8 m, at uniform speeds. One car travels at a speed of 20 cm/s faster than the other and reaches B two seconds earlier. How long does each car take to reach B?
12. S varies partly as v and partly as v^2. If S = 31 when v = 20 and S = 58 when v = 30, find the values of S when v = 25.
13. Expand $(a - x)^9$ up to the term in x^3. Use your expansion to estimate the value of $v(2.99)^9$ to 3 decimal places.
14. The electrical resistance of a wire of circular cross-section varies directly as the length and inversely as the square of the radius. Two wires have equal resistances and the radius of one is twice the radius of the other. Find the ratio of their lengths.
15. The distance between Kamau's home and school is 5 km to the nearest 10 m. Kamau can walk at 3 kmh^{-1} to the nearest 0.5 kmh^{-1}. Calculate the range of times Kamau can take for the trip to school.
16. (a) Find the constant term in the expansion of $(x + \frac{1}{x})^8$.
    (b) Use binomial expansion to approximate the value of $(0.98)^3$ to 4 decimal places.